The geometry of density states, positive maps and tomograms

V.I. Man'ko,[1] G. Marmo,[2] E.C.G. Sudarshan[3] and F. Zaccaria[2]
[1] *P.N. Lebedev Physical Institute, Leninskii Prospect, 53, Noscow 119991 Russia*
[2] *Dipartimento di Scienze Fisiche, Università "Federico II" di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte Sant Angelo, Via Cintia, I-80126 Napoli, Italy*
[3] *Physics Department, Center for Particle Physics, University of Texas, 78712 Austin, Texas, USA*
Emails **manko@sci.lebedev.ru marmo@na.infn.it sudarshan@physics.utexas.edu zaccaria@na.infn.it**

Abstract. The positive and not completely positive maps of density matrices, which are contractive maps, are discussed as elements of a semigroup. A new kind of positive map (the purification map), which is nonlinear map, is introduced. The density matrices are considered as vectors, linear maps among matrices are represented by superoperators given in the form of higher dimensional matrices. Probability representation of spin states (spin tomography) is reviewed and $U(N)$-tomogram of spin states is presented. Properties of the tomograms as probability distribution functions are studied. Notion of tomographic purity of spin states is introduced. Entanglement and separability of density matrices are expressed in terms of properties of the tomographic joint probability distributions of random spin projections which depend also on unitary group parameters. A new positivity criterion for hermitian matrices is formulated. An entanglement criterion is given in terms of a function depending on unitary group parameters and semigroup of positive map parameters. The function is constructed as sum of moduli of $U(N)$-tomographic symbols of the hermitian matrix obtained after action on the density matrix of composite system by a positive but not completely positive map of the subsystem density matrix. Some two-qubit and two-qutritt states are considered as examples of entangled states. The connection with the star-product quantisation is discussed. The structure of the set of density matrices and their relation to unitary group and Lie algebra of the unitary group are studied. Nonlinear quantum evolution of state vector obtained by means of applying purification rule of density matrices evolving via dynamical maps is considered. Some connection of positive maps and entanglement with random matrices is discussed and used.

Keywords: unitary group, entanglement, adjoint representation, tomogram, operator symbol, random matrix.

1. Introduction

The states in quantum mechanics are associated with vectors in Hilbert space [1] (it is better to say with rays) in the case of pure states. For mixed state, one associates the state with density matrix [2, 3].

In classical mechanics (statistical mechanics), the states are associated with joint probability distributions in phase space. There is an essential difference in the concept of states in classical and quantum mechanics. This difference is clearly pointed out by the phenomenon of entanglement. The notion of entanglement [4] is related to the quantum superposition rule of the states of subsystems for a given multipartite system. For pure states, the notion of entanglement and separability can be given as follows.

If the wave function [5] of a state of a bipartite system is represented as the product of two wave functions depending on coordinates of the subsystems, the state is simply separable; otherwise, the state is simply entangled. An intrinsic approach to the entanglement measure was suggested in [6]. The measure was introduced as the distance between the system density matrix and the tensor product of the associated states. For the subsystems, the association being realized via partial traces. There are several other different characteristics and measures of entanglement considered by several authors [7–13]. For example, there are measures related to entropy (see, [14–24]). Also linear entropy of entanglement was used in [25–27], "concurrences" in [28, 29] and "covariance entanglement measure" in [30]. Each of the entanglement measures describes some degree of correlation between the subsystems' properties.

The notion of entanglement is not an absolute notion for a given system but depends on the decomposition into subsystems. The same quantum state can be considered as entangled, if one kind of division of the system into subsystems is given, or as completely disentangled, if another decomposition of the system into subsystems is considered.

For instance, the state of two continuous quadratures can be entangled in Cartesian coordinates and disentangled in polar coordinates. Coordinates are considered as measurable observables labelling the subsystems of the given system. The choice of different subsystems mathematically implies the existence of two different sets of the subsystems' characteristics (we focus on bipartite case). We may consider the Hilbert space of states $H(1,2)$ or $H(1',2')$. The Hilbert space for the total system is, of course, the same but the index $(1,2)$ means that there are two sets of operators P_1 and P_2, which select subsystem states 1 and 2. The index $(1',2')$ means that there are two other sets of operators P_1' and P_2', which select subsystem states $1'$ and $2'$. The operators $P_{1,2}$ and $P_{1',2'}'$ have specific properties. They are represented as tensor products of operators acting in the space of states of the subsystem 1 (or 2) and unit operators acting in the subsystem 2 (or 1). In other words, we consider the space H, which can be treated as the tensor product of spaces $H(1)$ and $H(2)$ or $H(1')$ and $H(2')$. In

the subsystems 1 and 2, there are basis vectors $\mid n_1\rangle$ and $\mid m_2\rangle$, and in the subsystems $1'$ and $2'$ there are basis vectors $\mid n_1'\rangle$ and $\mid m_2'\rangle$. The vectors $\mid n_1\rangle \mid m_2\rangle$ and $\mid n_1'\rangle \mid m_2'\rangle$ form the sets of basis vectors in the composite Hilbert space, respectively. These two sets are related by means of unitary transformation. An example of such a composite system is a bipartite spin system.

If one has spin-j_1 [the space $H(1)$] and spin-j_2 [the space $H(2)$] systems, the combined system can be treated as having basis $\mid j_1m_1\rangle \mid j_2m_2\rangle$.

Another basis in the composite-system-state space can be considered in the form $\mid jm\rangle$, where j is one of the numbers $|j_1-j_2|, |j_1-j_2|+1, \ldots, j_1+j_2$ and $m = m_1 + m_2$. The basis $\mid jm\rangle$ is related to the basis $\mid j_1m_1\rangle \mid j_2m_2\rangle$ by means of the unitary transform given by Clebsch–Gordon coefficients $C\,(j_1m_1j_2m_2|jm)$. From the viewpoint of the given definition, the states $\mid jm\rangle$ are entangled states in the original basis. Another example is the separation of the hydrogen atom in terms of parabolic coordinates used while discussing the Stark effect.

The spin states can be described by means of the tomographic map [31–33]. For bipartite spin systems, the states were described by the tomographic probabilities in [34, 35]. Some properties of the tomographic spin description were studied in [36]. In the tomographic approach, the problems of the quantum state entanglement can be cast into the form of some relations among the probability distribution functions. On the other hand, to have a clear picture of entanglement, one needs a mathematical formulation of the properties of the density matrix of the composite system, a description of the linear space of the composite system states. Since a density matrix is hermitian, the space of states may be embedded as a subset of the Lie algebra of the unitary group, carrying the adjoint representation of $U(n^2)$, where $n^2 = (2j+1)^2$ is the dimension of the spin states of two spinning particles. Thus one may try to characterize the entanglement phenomena by using various structures present in the space of the adjoint representation of the $U(n^2)$ group.

The aim of this paper is to give a review of different aspects of density matrices and positive maps and connect entanglement problems with the properties of tomographic probability distributions and discuss the properties of the convex set of positive states for composite system by taking into account the subsystem structures. We used [6] the Hilbert–Schmidt distance to calculate the measure of entanglement as the distance between a given state and the tensor product of the partial traces of the density matrix of the given state. In [37] another measure of entanglement as a characteristic of subsystem correlations was introduced. This measure is determined via the covariance matrix of some

observables. A review of different approaches to the entanglement notion and entanglement measures is given in [38] where the approach to describe entanglement and separability of composite systems is based, e.g., on entropy methods.

Due to a variety of approaches to the entanglement problem, one needs to understand better what in reality the word "entanglement" describes. Is it a synonym of the word "correlation" between two subsystems or does it have to capture some specific correlations attributed completely and only to the quantum domain?

The paper is organized as follows.

In section 2 we discuss division of composite systems onto subsystems and relation of the density matrix to adjoint representation of unitary group in generic terms of vector representation of matrices; we study also completely positive maps of density matrices. In section 3 we consider a vector representation of probability distribution functions and notion of distance between the probability distributions and density matrices. In section 4 we present definition of separable quantum state of a composite system and criterion of separability. In section 5 the entanglement is considered in terms of operator symbols. Particular tomographic probability representation of quantum states and tomographic symbols are reviewed in section 6. Symbols of multipartite states are studied in section 7. In section 8 spin tomography is reviewed. An example of qubit state is done in section 9. The unitary spin tomogram is introduced in section 10 while in section 11 dynamical map and corresponding quantum evolution equations are discussed as well as examples of concrete positive maps. Conclusions and perspectives are presented in section 12.

2. Composite system

In this section, we review the meaning and notion of composite system in terms of additional structures on the linear space of state for the composite system.

2.1. Difference of states and observables

In quantum mechanics, there are two basic aspects, which are associated with linear operators acting in a Hilbert space. The first one is related to the concept of quantum state and the second one, to the concept of observable. These two concepts of state and observable are paired via a map with values in probability measures on the real line. Often states are described by Hermitian nonnegative, trace-class, matrices. The observables are described by Hermitian operators. Though

both states and observables are identified with Hermitian objects, there is an essential difference between the corresponding objects. The observables have an additional product structure. Thus we may consider the product of two linear operators corresponding to observables.

For the states, the notion of product is redundant. The product of two states is not a state. For states, one keeps only the linear structure of vector space. For finite n-dimensional system, the Hermitian states and the Hermitian observables may be mapped into the Lie algebra of the unitary group $U(n)$. But the states correspond to nonnegative Hermitian operators. Observables can be associated with both types of operators, including nonnegative and nonpositive ones. The space of states is therefore a convex-linear space which, in principle, is not equipped with a product structure. Due to this, transformations in the linear space of states need not preserve any product structure. In the set of observables, one has to be concerned with what is happening with the product of operators when some transformations are performed. State vectors can be transformed into other state vectors. Density operators also can be transformed. We will consider linear transformations of the density operators. The density operator has nonnegative eigenvalues. In any representation, diagonal elements of density matrix have physical meaning of probability distribution function.

Density operator can be decomposed as a sum of eigenprojectors with coefficients equal to its eigenvalues. Each one of the projectors defines a pure state. There exists a basis in which every eigenprojector of rank one is represented by a diagonal matrix of rank one with only one matrix element equal to one and all other matrix elements equal to zero. Other density matrices with similar properties belong to the orbit of the unitary group on the starting eigenprojector. Depending on the number of distinct nonzero values determines the class of the orbit. Since density matrices of higher rank belong to an appropriate orbit of a convex sum of the different diagonal eigenprojectors (in special basis), we may say that generic density matrices belong to the orbits of the unitary group acting on the diagonal density matrices which belong to the Cartan subalgebra of the Lie algebra of the unitary group. Any convex sum of density matrices can be treated as a mean value of a random density matrix. The positive coefficients of the convex sum can be interpreted as a probability distribution function which makes the averaging providing the final value of the convex sum. The set of density matrices may be identified with the union of the orbits of the unitary group acting on diagonal density matrices considered as elements of the Cartan subalgebra.

2.2. Matrices as vectors, density operators and superoperators

When matrices represent states it may be convenient to identify them with vectors. In this case, a density matrix can be considered as a vector with additional properties of its components. If the identifications are done elegantly, we can see the real Hilbert space of density matrices in terms of vectors with real components. In this case, linear transforms of the matrix can be interpreted as matrices called superoperators. It means that density matrices–vectors undergoing real linear transformations are acted on by the matrices representing the action of the superoperators of the linear map. This construction can be continued. Thus we can get a chain of vector spaces of higher and higher dimensions. Let us first introduce some extra constructions of the map of a matrix onto a vector. Given a rectangular matrix M with elements M_{id}, where $i = 1, 2, \ldots, n$ and $d = 1, 2, \ldots, m$, one can consider the matrix as a vector $\vec{\mathcal{M}}$ with $N = nm$ components constructed by the following rule:

$$\begin{aligned} &\mathcal{M}_1 = M_{11}, \qquad \mathcal{M}_2 = M_{12}, \qquad \mathcal{M}_m = M_{1m}, \\ &\mathcal{M}_{m+1} = M_{21}, \ldots \mathcal{M}_N = M_{nm}. \end{aligned} \tag{1}$$

Thus we construct the map $M \to \vec{\mathcal{M}} = \hat{t}_{\vec{\mathcal{M}}M} M$.

We have introduced the linear operator $\hat{t}_{\vec{\mathcal{M}}M}$ which maps the matrix M onto a vector $\vec{\mathcal{M}}$. Now we introduce the inverse operator $\hat{p}_{\vec{\mathcal{M}}M}$ which maps a given column vector in the space with dimension $N = mn$ onto a rectangular matrix. This means that given a vector $\vec{\mathcal{M}} = \mathcal{M}_1, \ldots, \mathcal{M}_N$, we relabel its components by introducing two indices $i = 1, \ldots, n$ and $d = 1, \ldots, m$. The relabeling is accomplished according to (1). Then we collect the relabeled components into a matrix table. Eventually we get the map

$$\hat{p}_{\vec{\mathcal{M}}M} \vec{\mathcal{M}} = M. \tag{2}$$

The composition of these two maps

$$\hat{t}_{\vec{\mathcal{M}}M} \hat{p}_{\vec{\mathcal{M}}M} \vec{\mathcal{M}} = 1 \cdot \vec{\mathcal{M}} \tag{3}$$

acts as the unit operator in the linear space of vectors.

Given a $n \times n$ matrix the map considered can also be applied. The matrix can be treated as an n^2-dimensional vector and, vice versa, the vector of dimension n^2 may be mapped by this procedure onto the $n \times n$ matrix.

Let us consider a linear operator acting on the vector $\vec{\mathcal{M}}$ and related to a linear transform of the matrix M. First, we study the correspondence of the linear transform of the form

$$M \to gM = M_g^l \tag{4}$$

to the transform of the vector

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}_g^l = \mathcal{L}_g^l \vec{\mathcal{M}}. \tag{5}$$

One can show that the $n^2 \times n^2$ matrix $\mathcal{L}_g^l$ is determined by the tensor product of the $n \times n$ matrix g and $n \times n$ unit matrix, i.e.,

$$\mathcal{L}_g^l = g \otimes 1. \tag{6}$$

Analogously, the linear transform of the matrix M of the form

$$M \to Mg = M_g^r \tag{7}$$

induces the linear transform of the vector $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}_g^r = \hat{t}_{\vec{\mathcal{M}}\mathcal{M}} M_g^r = \mathcal{L}_g^r \vec{\mathcal{M}}, \tag{8}$$

where the $n^2 \times n^2$ matrix $\mathcal{L}_g^r$ reads

$$\mathcal{L}_g^r = 1 \otimes g^{\mathrm{tr}}. \tag{9}$$

Similarity transformation of the matrix M of the form

$$M \to gMg^{-1} \tag{10}$$

induces the corresponding linear transform of the vector $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}_s = \mathcal{L}_g^s \vec{\mathcal{M}}, \tag{11}$$

where the $n^2 \times n^2$ matrix $\mathcal{L}_g^s$ reads

$$\mathcal{L}_g^s = g \otimes (g^{-1})^{\mathrm{tr}}. \tag{12}$$

Starting with vectors, one may ask how to identify on them a product structure which would make $\hat{p}_{\vec{\mathcal{M}}\mathcal{N}}$ into an algebra homomorphism. An associative algebraic structure on the vector space may be defined by imitating the procedure one uses to define star-products on the space of functions on phase space. One can define the associative product of two N-vectors $\vec{\mathcal{M}}_1$ and $\vec{\mathcal{M}}_2$ using the rule

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}_1 \star \vec{\mathcal{M}}_2, \tag{13}$$

where

$$\vec{\mathcal{M}}_k = \sum_{l,s=1}^{N} K_{ls}^k (\vec{\mathcal{M}}_1)_l (\vec{\mathcal{M}}_2)_s. \tag{14}$$

If one applies a linear transform to the vectors $\vec{\mathcal{M}}_1$, $\vec{\mathcal{M}}_2$, $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}}_1 \to \vec{\mathcal{M}}_1' = \mathcal{L}\vec{\mathcal{M}}_1, \qquad \vec{\mathcal{M}}_2 \to \vec{\mathcal{M}}_2' = \mathcal{L}\vec{\mathcal{M}}_2, \qquad \vec{\mathcal{M}} \to \vec{\mathcal{M}}' = \mathcal{L}\vec{\mathcal{M}},$$

and requires the invariance of the star-product kernel, one finds

$$\vec{\mathcal{M}}_1' \star \vec{\mathcal{M}}_2' = \vec{\mathcal{M}}', \qquad \text{if} \qquad \mathcal{L} = G \otimes G^{-1\mathrm{tr}}, \quad G \in GL(n).$$

The kernel K_{ls}^k (structure constants) which determines the associative star-product satisfies a quadratic equation. Thus if one wants to make the correspondence of the vector star-product to the standard matrix product (row by column), the matrix M must be constructed appropriately. For example, if the vector star-product is commutative, the matrix M corresponding to the N-vector $\vec{\mathcal{M}}$ can be chosen as a diagonal $N \times N$ matrix. This consideration shows that the map of matrices on the vectors provides the star-product of the vectors (defining the structure constants or the kernel of the star-product) and, conversely, if one starts with vectors and uses matrices with the standard multiplication rule, it will be the map to be determined by the structure constants (or by the kernel of the vector star-product).

The constructed space of matrices associated with vectors enables one to enlarge the dimensionality of the group acting in the linear space of matrices in comparison with the standard one, i.e., we may relax the requirement of invariance of the product structure. In general, given a $n \times n$ matrix M the left action, the right action, and the similarity transformation of the matrix are related to the complex group $GL(n)$. On the other hand, the linear transformations in the linear space of n^2-vectors $\vec{\mathcal{M}}$ obtained by using the introduced map are determined by the matrices belonging to the group $GL(n^2)$. There are transformations on the vectors which cannot be simply represented on matrices. If $M \to \Phi(M)$ is a linear homogeneous function of the matrix M, we may represent it by

$$\Phi_{ab} = B_{aa',bb'} M_{a'b'}.$$

Under rather clear conditions, $B_{aa',bb'}$ can be expressed in terms of its nonnormalized left and right eigenvectors:

$$B_{aa',bb'} = \sum_{\nu} x_{aa'}(\nu) y^{\dagger}_{bb'}(\nu),$$

being an index for eigenvalues, which corresponds to

$$\Phi(M) = xMy^{\dagger} = \sum_{\nu=1}^{n^2} x(\nu) M y^{\dagger}(\nu).$$

There are possible linear transforms on the matrices and corresponding linear transforms on the induced vector space which do not give rise to a group structure but possess only the structure of algebra. One can describe the map of $n\times n$ matrices M (source space) onto vectors $\vec{\mathcal{M}}$ (target space) using specific basis in the space of the matrices. The basis is given by the matrices E_{jk} $(j,k=1,2,\ldots,n)$ with all matrix elements equal to zero except the element in the jth row and kth column which is equal to unity. One has the obvious property

$$M_{jk}=\mathrm{Tr}\left(ME_{jk}\right). \tag{15}$$

In our procedure, the basis matrix E_{jk} is mapped onto the basis column-vector $\vec{\mathcal{E}}_{jk}$, which has all components equal to zero except the unity component related to the position in the matrix determined by the numbers j and k. Then one has

$$\vec{\mathcal{M}}=\sum_{j,k=1}^{n}\mathrm{Tr}\left(ME_{jk}\right)\vec{\mathcal{E}}_{jk}. \tag{16}$$

For example, for similarity transformation of the finite matrix M, one has

$$\vec{\mathcal{M}}_g^s=\sum_{j,k=1}^{N}\mathrm{Tr}\left(gMg^{-1}E_{jk}\right)\vec{\mathcal{E}}_{jk}. \tag{17}$$

Now we will define the notion of 'composite' vector which corresponds to dividing a quantum system into subsystems.

We will use the following terminology.

In general, the given linear space of dimensionality $N=mn$ has a structure of a bipartite system, if the space is equipped with the operator $\hat{p}_{\vec{\mathcal{M}}M}$ and the matrix M (obtained by means of the map) has matrix elements in factorizable form

$$M_{id}\to x_i y_d. \tag{18}$$

This $M=x\otimes y$ corresponds to the special case of nonentangled states. Otherwise, one needs

$$M=\sum_{\nu}x(\nu)\otimes y(\nu).$$

In fact, to consider in detail the entanglement phenomenon, in the bipartite system of spin-1/2, one has to introduce a hierarchy of three linear spaces. The first space of pure spin states is the two-dimensional linear space of complex vectors

$$\mid\vec{x}\rangle=\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{19}$$

In this space, the scalar product is defined as follows:

$$\langle \vec{x} \mid \vec{y} \rangle = x_1^* y_1 + x_2^* y_2. \tag{20}$$

So it is a two-dimensional Hilbert space. We do not equip this space with a vector star-product structure. In the primary linear space, one introduces linear operators $\hat{M}$ which are described by 2×2 matrices M. Due to the map discussed in the previous section, the matrices are represented by 4-vectors $\vec{\mathcal{M}}$ belonging to the second complex 4-dimensional space. The star-product of the vectors $\vec{\mathcal{M}}$ determined by the kernel $\mathcal{K}_{ls}^k$ is defined in such a manner in order to correspond to the standard rule of multiplication of the matrices.

In addition to the star-product structure, we introduce the scalar product of the vectors $\vec{\mathcal{M}}_1$ and $\vec{\mathcal{M}}_2$, in view of the definition

$$\langle \vec{\mathcal{M}}_1 \mid \vec{\mathcal{M}}_2 \rangle = \mathrm{Tr}\,(M_1^\dagger M_2), \tag{21}$$

which is the trace formula for the scalar product of matrices.

This means introducing the real metric $g^{\alpha\beta}$ in the standard notation for scalar product

$$\langle \vec{\mathcal{M}}_1 \mid \vec{\mathcal{M}}_2 \rangle = \sum_{\alpha,\beta=1}^{4} (\mathcal{M}_1)_\alpha^* g^{\alpha\beta} (\mathcal{M}_2)_\beta, \tag{22}$$

where the matrix $g^{\alpha\beta}$ is of the form

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{\alpha j} g^{j\beta} = \delta^{\alpha\beta}. \tag{23}$$

The scalar product is invariant under the action of the group of non-singular 4×4 matrices ℓ, which satisfy the condition

$$g = \ell^\dagger g \ell. \tag{24}$$

The product of matrices ℓ satisfies the same condition since $g^2 = 1$.

Thus, the space of operators $\hat{M}$ in the primary two-dimensional space of spin states is mapped onto the linear space which is equipped with a scalar product (metric Hilbert space structure) and an associative star-product (kernel satisfying the quadratic associativity equation). In the linear space of the 4-vectors $\vec{\mathcal{M}}$, we introduce linear operators (superoperators), which can be associated with the algebra of 4×4 complex matrices.

Let us now focus on density matrices. This means that our matrix M is considered as a density matrix ρ which describes a quantum state.

We consider here the action of the unitary transformation $U(n)$ of the density matrices and corresponding transformations on the vector space. If one has the structure of a bipartite system, we also consider the action of local gauge transformation both in the "source space" of density matrices and in the "target space" of the corresponding vectors.

The $n\times n$ density matrix ρ has matrix elements

$$\rho_{ik} = \rho^*_{ki}, \qquad \mathrm{Tr}\,\rho = 1, \qquad \langle\psi|\rho|\psi\rangle \geq 0. \tag{25}$$

Since the density matrix is hermitian, it can always be identified as an element of the convex subset of the linear space associated with the Lie algebra of $U(n)$ group, on which the group $U(n)$ acts with the adjoint representation

$$\rho \to \rho_U = U\rho U^\dagger. \tag{26}$$

The system is said to be bipartite if the space of representation is equipped with an additional structure. This means that for

$$n^2 = n_1 \cdot n_2,$$

where, for simplicity, $n_1 = n_2 = n$, one can make first the map of $n\times n$ matrix ρ onto n^2-dimensional vector $\vec{\rho}$ according to the previous procedure, i.e., one equips the space by an operator $\hat{t}_{\vec{\rho}\rho}$. Given this vector one makes a relabeling of the vector $\vec{\rho}$ components according to the rule

$$\vec{\rho} \to \rho_{id,ke}, \quad i,k = 1,2,\ldots,n_1, \quad d,e = 1,2,\ldots,n_2, \tag{27}$$

i.e., obtaining again the quadratic matrix

$$\rho_q = \hat{p}_{\rho_q\vec{\rho}}\vec{\rho}. \tag{28}$$

The unitary transform (26) of the density matrix induces a linear transform of the vector $\vec{\rho}$ of the form

$$\vec{\rho} \to \vec{\rho}_U = (U \otimes U^*)\vec{\rho}. \tag{29}$$

There exist linear transforms (called positive maps) of the density matrix, which preserve its trace, hermicity, and positivity. In some cases, they have the following form introduced in [39]

$$\rho_0 \to \rho_U = L_U\rho_0 = \sum_k p_k U_k \rho_0 U_k^\dagger, \qquad \sum_k p_k = 1, \tag{30}$$

where U_k are unitary matrices and p_k are positive numbers.

If the initial density matrix is diagonal, i.e., it belongs to the Cartan subalgebra of the Lie algebra of the unitary group, the diagonal elements of the obtained matrix give a "smoother" probability distribution

than the initial one. A generic transformation preserving previously stated properties may be given in the form (see [39, 40])

$$\rho_0 \to \rho_V = L_V \rho_0 = \sum_k V_k \rho_0 V_k^\dagger, \qquad \sum_k V_k^\dagger V_k = 1. \tag{31}$$

For example, if V_k $(k = 1, 2, \ldots, N)$ are taken as square roots of orthogonal projectors onto complete set of N state, the map provides the map of the density matrix ρ_0 onto diagonal density matrix ρ_{0d} which has the same diagonal elements as ρ_0 has. In this case, the matrices V_k have the only nonzero matrix element which is equal to one. Such a map may be called "decoherence map" because it removes all nondiagonal elements of the density matrix ρ_0 killing any phase relations. In quantum information terminology, one uses also the name "phase damping channel." More general map may be given if one takes V_k as N generic diagonal density matrices, in which eigenvalues are obtained by N circular permutations from the initial one. Due to this map, one has a new matrix with the same diagonal matrix elements but with changed nondiagonal elements. The purity of this matrix is smaller then the purity of the initial one. This means that the map is contractive. All matrices with the same diagonal elements up to permutations belong to a given orbit of the unitary group.

For a large number of terms with randomly chosen matrices V_k in the sum in (31), the above map gives the most stochastic density matrix

$$\rho_0 \to \rho_s = L_1 \rho_0 = (n)^{-1} 1.$$

Its four-dimensional matrix L_1 for the qubit case has four matrix elements different from zero. These matrix elements are equal to one. They have the labels L_{11}, L_{14}, L_{41}, L_{44}. The map with two nonzero matrix elements $L_{41} = L_{44} = 0$ provides pure-state density matrix from any ρ_0. The transform (30) is the partial case of the transform (31). We discuss the transforms separately since they are used in the literature in the presented form.

One can see that the constructed map of density matrices onto vectors provides the corresponding transforms of the vectors, i.e.,

$$\vec{\rho}_0 \to \vec{\rho}_U = \sum_k p_k (U_k \otimes U_k^*) \vec{\rho}_0 \tag{32}$$

and

$$\vec{\rho}_0 \to \vec{\rho}_V = \sum_k (V_k \otimes V_k^*) \vec{\rho}_0. \tag{33}$$

It is obvious that the set of linear transforms of vectors, which preserve their properties of being image of density matrices, is essentially larger than the standard unitary transform of the density matrices.

Formulae (32) and (33) mean that the positive map superoperators acting on the density matrix in the vector representation are described by $n^2 \times n^2$ matrices

$$\mathcal{L}_U = \sum_k p_k (U_k \otimes U_k^*) \tag{34}$$

and

$$\mathcal{L}_V = \sum_k V_k \otimes V_k^*, \tag{35}$$

respectively.

The positive map is called "noncompletely positive" if

$$\mathcal{L} = \sum_k V_k \otimes V_k^* - \sum_s v_s \otimes v_s^*, \qquad \sum_k V_k^\dagger V_k - \sum_s v_s^\dagger v_s = 1.$$

This map is related to a possible "nonphysical" evolution of a subsystem.

Formula (34) can be considered in the context of random matrix representation. In fact, the matrix $\mathcal{L}_U$ can be interpreted as the weighted mean value of the random matrix $U_k \otimes U_k^*$. The dependence of matrix elements and positive numbers p_k on index k means that we have a probability distribution function p_k and averaging of the random matrix $U_k \otimes U_k^*$ by means of the distribution function. So the matrix $\mathcal{L}_U$ reads

$$\mathcal{L}_U = \langle U \otimes U^* \rangle. \tag{36}$$

Let us consider an example of a 2×2 unitary matrix. We can consider a matrix of the $SU(2)$ group of the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{37}$$

The 4×4 matrix $\mathcal{L}_U$ takes the form

$$\mathcal{L}_U = \begin{pmatrix} \ell & m & m^* & 1-\ell \\ -n & s & -q & n \\ -n^* & -q^* & s^* & n^* \\ 1-\ell & -m & -m^* & \ell \end{pmatrix}. \tag{38}$$

The matrix elements of the matrix $\mathcal{L}_U$ are the means

$$\begin{aligned} m &= \langle \alpha\beta^* \rangle, \\ \ell &= \langle \alpha\alpha^* \rangle, \\ n &= \langle \alpha\beta \rangle, \\ s &= \langle \alpha^2 \rangle, \\ q &= \langle \beta^2 \rangle. \end{aligned} \tag{39}$$

The moduli of these matrix elements are smaller than unity.

The determinant of the matrix $\mathcal{L}_U$ reads

$$\det \mathcal{L}_U = (1-2\ell)\Big(|q|^2 - |s|^2\Big) + 4\,\mathrm{Re}\Big[q^* m^* n + mns^*\Big]. \tag{40}$$

If one represents the matrix $\mathcal{L}_U$ in block form

$$\mathcal{L}_U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{41}$$

then

$$A = \begin{pmatrix} \ell & m \\ -n & s \end{pmatrix}, \qquad B = \begin{pmatrix} m^* & 1-\ell \\ -q & n \end{pmatrix}, \tag{42}$$

and

$$D = \sigma_2 A^* \sigma_2, \qquad C = -\sigma_2 B^* \sigma_2, \tag{43}$$

where σ_2 is the Pauli matrix.

One can check that the product of two different matrices $\mathcal{L}_U$ can be cast in the same form. This means that the matrices $\mathcal{L}_U$ form a 9-parameter compact semigroup. It means that the product of two matrices from the set (semigroup) belongs to the same set. It means that composition is inner like the one for groups. There is a unity element in the semigroup, however, there exist elements which have no inverse. In our case, these elements are described, e.g., by the matrices with zero determinant. Also the elements, which are matrices with nonzero determinants, have no inverse elements in this set, since the map corresponding to the inverse of these matrices is not positive one. For example, in the case $\ell = 1/2$ and $m = 0$, one has the matrices

$$A = \begin{pmatrix} 1/2 & 0 \\ -n & s \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1/2 \\ -q & n \end{pmatrix}. \tag{44}$$

The determinant of the matrix $\mathcal{L}_U$ in this case is equal to zero. All the matrices $\mathcal{L}_U$ have the eigenvector

$$\vec{\rho}_0 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}, \tag{45}$$

i.e.,

$$\mathcal{L}_U \vec{\rho}_0 = \vec{\rho}_0. \tag{46}$$

This eigenvector corresponds to the density matrix

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \tag{47}$$

which is obviously invariant of the positive map.

For random matrix, one has correlations of the random matrix elements, e.g., $\langle \alpha\alpha^* \rangle \neq \langle \alpha \rangle \langle \alpha^* \rangle$.

The matrix $\mathcal{L}_p$

$$\mathcal{L}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

maps the vector

$$\vec{\rho}_{\rm in} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} \tag{49}$$

onto the vector

$$\vec{\rho}_{\rm t} = \begin{pmatrix} \rho_{11} \\ \rho_{21} \\ \rho_{12} \\ \rho_{22} \end{pmatrix}. \tag{50}$$

This means that the positive map (48) connects the positive density matrix with its transpose (or complex conjugate). This map can be presented as the connection of the matrix ρ with its transpose of the form

$$\rho \to \rho^{\rm tr} = \rho^* = \frac{1}{2}\Big(\rho + \sigma_1\rho\sigma_1 - \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3\Big).$$

There is no unitary transform connecting these matrices.

There is noncompletely positive map in the N-dimensional case, which is given by the generalized formula (for some ε)

$$\rho \to \rho_s = -\varepsilon\rho + \frac{1+\varepsilon}{N}\,1_N.$$

In quantum information terminology, it is called "depolarizing channel."

For the qubit case, matrix form of this map reads

$$L = \begin{pmatrix} \frac{1-\varepsilon}{2} & 0 & 0 & \frac{1+\varepsilon}{2} \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ \frac{1+\varepsilon}{2} & 0 & 0 & \frac{1-\varepsilon}{2} \end{pmatrix}. \tag{51}$$

Thus we constructed the matrix representation of the positive map of density operators of the spin-1/2 system. This particular set of matrices realize the representation of the semigroup of real numbers $-1 \leq \varepsilon \leq 1$. If one considers the product $\varepsilon_1\varepsilon_2 = \varepsilon_3$, the result ε_3 belongs to the

semigroup. Only two elements 1 and -1 of the semigroup have the inverse. These two elements form the finite subgroup of the semigroup. The semigroup itself without element $\varepsilon = 0$ can be embedded into the group of real numbers with natural multiplication rule. Each matrix L has an inverse element in this group but all the parameters of the inverse elements η live out of the segment $-1, 1$. The group of the real numbers is commutative. The matrices of the nonunitary representation of this group commute too. It means that we have nonunitary reducible representation of the semigroup which is also commutative. To construct this representation, one needs to use the map of matrices on the vectors discussed in the previous section. Formulae (31) and (35) can be interpreted also in the context of the random matrix representation, but we use the uniform distribution for averaging in this case. So one has equality (35) in the form

$$\mathcal{L}_V = \langle V \otimes V^* \rangle \tag{52}$$

and the equality

$$\langle V^\dagger V \rangle = 1, \tag{53}$$

which provides constraints for the random matrices V used.

Using the random matrix formalism, the positive (but not completely positive) maps can be presented in the form

$$\mathcal{L} = \langle V \otimes V^* \rangle - \langle v \otimes v^* \rangle, \qquad \langle V^\dagger V \rangle - \langle v^\dagger v \rangle = 1.$$

One can characterize the action of positive map on a density matrix ρ by the parameter

$$\kappa = \frac{\mathrm{Tr}\,(\mathcal{L}\rho)^2}{\mathrm{Tr}\,\rho^2} = \frac{\mu_{\mathcal{L}\rho}}{\mu_\rho} \leq 1.$$

As a remark we note that in [39] the positive maps (30) and (31) were used to describe the non-Hamiltonian evolution of quantum states for open systems.

We have to point out that, in general, such evolution is not described by first-order-in-time differential equation. As in the previous case, if there are additional structures for the matrix in the form

$$\rho_{id,ke} \to x_i y_d z_k t_e, \tag{54}$$

which means associating with the initial linear space two extra linear spaces where x_i, z_k are considered as vector components in the n_1-dimensional linear space and y_d, t_e are vector components in n_2-dimensional vector space, we see that one has bipartite structure of the initial space of state [bipartite structure of the space of adjoint representations of the group $U(n)$].

Usually the adjoint representation of any group is defined per se without any reference to possible substructures. Here we introduce the space with extra structure. In addition to being the space of the adjoint representation of the group $U(n)$, it has the structure of a bipartite system. The generalization to multipartite (N-partite) structure is straightforward. One needs only the representation of positive integer n^2 in the form

$$n^2 = \prod_{k=1}^{N} n_k^2. \tag{55}$$

If one considers the more general map given by superoperator (35) rewritten in the form

$$\mathcal{L}_V = \langle V \otimes V^* \rangle, \qquad \langle V^\dagger V \rangle = 1,$$

the number of parameters determining the matrix $\mathcal{L}_V$ can be easily evaluated. For example, for $n = 2$,

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad V^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix},$$

where the matrix elements are complex numbers, the normalization condition provides four constraints for the real and imaginary parts of the matrix elements of the following matrix:

$$\mathcal{L}_V = \begin{pmatrix} \langle |a|^2 \rangle & \langle ab^* \rangle & \langle ba^* \rangle & \langle bb^* \rangle \\ \langle ac^* \rangle & \langle ad^* \rangle & \langle bc^* \rangle & \langle bd^* \rangle \\ \langle ca^* \rangle & \langle cb^* \rangle & \langle da^* \rangle & \langle db^* \rangle \\ \langle cc^* \rangle & \langle cd^* \rangle & \langle dc^* \rangle & \langle dd^* \rangle \end{pmatrix},$$

namely,

$$\langle |a|^2 \rangle + \langle |c|^2 \rangle = 1, \qquad \langle |b|^2 \rangle + \langle |d|^2 \rangle = 1, \qquad \langle a^* b \rangle + \langle c^* d \rangle = 0.$$

Due to the structure of the matrix $\mathcal{L}_V$, there are six complex parameters

$$\langle ab^* \rangle, \quad \langle ac^* \rangle, \quad \langle ad^* \rangle, \quad \langle bc^* \rangle, \quad \langle bd^* \rangle, \quad \langle cd^* \rangle$$

or 12 real parameters.

The geometrical picture of the positive map can be clarified if one considers the transform of the positive density matrix into another density matrix as the transform of an ellipsoid into another ellipsoid. A generic positive transform means a generic transform of the ellipsoid, which changes its orientation, values of semiaxis, and position in the space. But the transform does not change the ellipsoidal surface into a hyperboloidal or paraboloidal surface. For pure states, the positive density matrix defines the quadratic form which is maximally degenerated.

In this sense, the "ellipsoid" includes all its degenerate forms corresponding to the density matrix of rank less than n (in n-dimensional case). The number of parameters defining the map $\langle V \otimes V^* \rangle$ in the n-dimensional case is equal to $n^2(n^2-1)$.

The linear space of Hermitian matrices is also equipped with the commutator structure defining the Lie algebra of the group $U(n)$. The kernel that defines this structure (Lie product structure) is determined by the kernel that determines the star-product.

3. Distributions as vectors

In quantum mechanics, one needs the concept of distance between the quantum states. In this section, we consider the notion of distance between the quantum states in terms of vectors. First, let us discuss the notion of distance between conventional probability distributions. This notion is well known in the classical probability theory.

Given the probability distribution $P(k)$, $k = 1, 2, \dots N$, one can introduce the vector $\vec{P}$ in the form of a column with components $P_1 = P(1)$, $P_2 = P(2), \dots$, $P_N = P(N)$. The vector satisfies the condition

$$\sum_{k=1}^{N} P_k = 1. \tag{56}$$

This set of vectors does not form a linear space but only a convex subset. Nevertheless, in this set one can introduce a distance between two distributions by using the one suggested by the vector space structure of the ambient space:

$$D^2 = \left(\vec{P}_1 - \vec{P}_2\right)^2 = \sum_k P_{1k} P_{1k} + \sum_k P_{2k} P_{2k} - 2 \sum_k P_{1k} P_{2k}. \tag{57}$$

Of course, one may use other identifications of distributions with vectors.

Since all $P(k) \geq 0$, one can introduce $\mathcal{P}_k = \sqrt{P(k)}$ as components of the vector $\vec{\mathcal{P}}$. The $\vec{\mathcal{P}}$ can be thought of as a column with nonnegative components. Then the distance between the two distributions takes the form

$$\mathcal{D}^2 = \left(\vec{\mathcal{P}}_1 - \vec{\mathcal{P}}_2\right)^2 = 2 - 2 \sum_k \sqrt{P_1(k) P_2(k)}. \tag{58}$$

The two different definitions (56) and (57) can be used as distances between the distributions.

Let us discuss now the notion of distance between the quantum states determined by density matrices. In the density-matrix space (in

the set of linear space of the adjoint $U(n)$ representation), one can introduce distances analogously. The first case is

$$\mathrm{Tr}\,(\rho_1-\rho_2)^2=D^2 \tag{59}$$

and the second case is

$$\mathrm{Tr}\,(\sqrt{\rho_1}-\sqrt{\rho_2})^2=\mathcal{D}^2. \tag{60}$$

In fact, the distances introduced can be written naturally as norms of vectors associated to density matrices

$$D^2=|\vec{\rho}_1-\vec{\rho}_2|^2 \tag{61}$$

and

$$\mathcal{D}^2=\Big((\vec{\sqrt{\rho}}_1)-(\vec{\sqrt{\rho}}_2)\Big)^2, \tag{62}$$

respectively.

In the above expressions, we use scalar product of vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ as well as scalar products of vectors $(\vec{\sqrt{\rho}}_1)$ and $(\vec{\sqrt{\rho}}_2)$, respectively.

Both definitions immediately follow by identification of either matrices ρ_1 and ρ_2 with vectors according to the map of the previous sections or matrices $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ with vectors. Since the density matrices ρ_1 and ρ_2 have nonnegative eigenvalues, the matrices $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ are defined without ambiguity. This means that the vectors $(\vec{\sqrt{\rho}}_1)$ and $(\vec{\sqrt{\rho}}_2)$ are also defined without ambiguity. It is obvious that using this construction and introducing linear map of positive vectors $\vec{\sqrt{\rho}}$, one induces nonlinear map of density matrices. Other analogous functions, in addition to square root function, can be used to create other nonlinear positive maps.

4. Separable systems and separability criterion

According to the definition, the system density matrix is called separable (for composite system) but not simply separable, if there is decomposition of the form

$$\rho_{AB}=\sum_k p_k\Big(\rho_A^{(k)}\otimes\rho_B^{(k)}\Big),\qquad \sum_k p_k=1,\qquad 1\geq p_k\geq 0. \tag{63}$$

This is Hilbert's problem of biquadrates. Is a positive biquadratic the positive sum of products of positive quadratics? In this formula, one may use also sum over two different indices. Using another labelling in such sum over two different indices, this sum can be always represented

as the sum over only one index. The formula does not demand orthogonality of the density operators $\rho_A^{(k)}$ and $\rho_B^{(k)}$ for different k. Since every density matrix is a convex sum of pure density matrices, one could demand that $\rho_A^{(k)}$ and $\rho_B^{(k)}$ be pure. This formula can be interpreted in the context of random matrix representation reading

$$\rho_{AB} = \langle \rho_A \otimes \rho_B \rangle, \tag{64}$$

where ρ_A and ρ_B are considered as random density matrices of the subsystems A and B, respectively. One can use the clarified structure of the density matrix set as the union of orbits obtained by action of the unitary group on projectors of rank one with matrix form containing only one nonzero matrix element. Then the separable density matrix of bipartite composite system can be always written as the sum of $n_1 n_2$ tensor products (or corresponding mean tensor product), i.e., in (64) the factors are state projectors. Each of tensor products contains random unitary matrices of local transforms of the fixed local projector for one subsystem and for the second subsystem. It means that an arbitrary projector of rank one of a subsystem can be always presented in the product form $\rho_A^{(k)} = u_A^{(k)} \rho_A u_A^{(k)\dagger}$, where $u_A^{(k)}$ is a unitary local transform and ρ_A is a fixed projector.

There are several criteria for the system to be separable. We suggest in the next sections a new approach to the problem of separability and entanglement based on the tomographic probability description of quantum states. The states which cannot be represented in the form (63) by definition are called entangled states [38]. Thus the states are entangled if in formula (63) at least one coefficient (or more) p_i is negative which means that the positive ones can take values greater than unity.

Let us discuss the condition for the system state to be separable. According to the partial transpose criterion [41], the system is separable if the partial transpose of the matrix ρ_{AB} (63) gives a positive density matrix. This condition is necessary but not sufficient. Let us discuss this condition within the framework of positive-map matrix representation. For example, for a spin-1/2 bipartite system, we have shown that the map of a density matrix onto its transpose belongs to the matrix semigroup of matrices $\mathcal{L}$. One should point out that this map cannot be obtained by means of averaging with all positive probability distributions p_k. On the other hand, it is obvious that the generic criterion, which contains the Peres criterion as a partial case, can be formulated as follows.

Let us map the density matrix ρ_{AB} of a bipartite system onto vector $\vec{\rho}_{AB}$. Let the vector $\vec{\rho}_{AB}$ be acted upon by an arbitrary matrix, which

represents the positive maps in subsystems A and B. Thus we get a new vector

$$\vec{\rho}_{AB}^{(p)} = \Big(\mathcal{L}_A \otimes \mathcal{L}_B\Big)\vec{\rho}_{AB}. \tag{65}$$

Let us construct the density matrix $\rho_{AB}^{(p)}$ using the inverse map of the vectors onto matrices. If the initial density matrix is separable, the new density matrix $\rho_{AB}^{(p)}$ must be positive (and separable).

In the case of the bipartite spin-1/2 system, by choosing $\mathcal{L}_A = 1$ and with $\mathcal{L}_B$ being the matrix coinciding with the matrix $g^{\alpha\beta}$, we obtain the Peres criterion as a partial case of the criterion of separability formulated above. Thus, our criterion means that the separable matrix keeps positivity under the action of the tensor product of two semigroups. In the case of the bipartite spin-1/2 system, the 16×16 matrix of the semigroup tensor product of positive contractive maps (52) is determined by 24 parameters. Among these parameters, one can have some correlations.

Let us discuss the positive map (52) which is determined by the semigroup for the n-dimensional system. It can be realized also as follows.

The $n\times n$ Hermitian generic matrix ρ can be mapped onto essentially real n^2-vector $\vec{\rho}$ by the map described above. The complex vector $\vec{\rho}$ is mapped onto the real vector $\vec{\rho}_{\rm r}$ by multiplying by the unitary matrix S, i.e.,

$$\vec{\rho}_{\rm r} = S\vec{\rho}, \qquad \vec{\rho} = S^{-1}\vec{\rho}_{\rm r}. \tag{66}$$

The matrix S is composed from n unity blocks and the blocks

$$S_b^{(jk)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{67}$$

where j corresponds to a column and k corresponds to a row in the matrix ρ.

For example, in the case $n = 2$, one has the vector $\vec{\rho}_{\rm r}$ of the form

$$\vec{\rho}_{\rm r} = \begin{pmatrix} \rho_{11} \\ \sqrt{2}\,{\rm Re}\,\rho_{12} \\ \sqrt{2}\,{\rm Im}\,\rho_{12} \\ \rho_{22} \end{pmatrix}. \tag{68}$$

One has the equalities

$$\vec{\rho}_{\rm r}^{\,2} = \vec{\rho}^{\,2} = {\rm Tr}\,\rho^2. \tag{69}$$

The semigroup preserves the trace of the density matrix. Also the discrete transforms, which are described by the matrix with diagonal

matrix blocks of the form

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{70}$$

preserve positivity of the density matrix.

For the spin case, the semigroup contains 12 parameters.

Thus, the direct product of the semigroup (52) and the discrete group of the transform D defines positive map preserving positivity of the density operator. One can include also all the matrices which correspond to other not completely positive maps. The considered representation contains only real vectors and their real positive transforms. This means that one can construct representation of semigroup of positive maps by real matrices.

5. Symbols, star-product and entanglement

In this section, we describe how entangled states and separable states can be studied using properties of symbols and density operators of different kinds, e.g., from the viewpoint of the Wigner function or tomogram. The general scheme of constructing the operator symbols is as follows [36].

Given a Hilbert space H and an operator $\hat{A}$ acting on this space, let us suppose that we have a set of operators $\hat{U}(\mathbf{x})$ acting transitively on H parametrized by n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We construct the c-number function $f_{\hat{A}}(\mathbf{x})$ (we call it the symbol of the operator $\hat{A}$) using the definition

$$f_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right]. \tag{71}$$

Let us suppose that relation (71) has an inverse, i.e., there exists a set of operators $\hat{D}(\mathbf{x})$ acting on the Hilbert space such that

$$\hat{A} = \int f_{\hat{A}}(\mathbf{x})\hat{D}(\mathbf{x})\ d\mathbf{x}, \qquad \mathrm{Tr}\,\hat{A} = \int f_{\hat{A}}(\mathbf{x})\,\mathrm{Tr}\,\hat{D}(\mathbf{x})\ d\mathbf{x}. \tag{72}$$

One needs a measure in $\mathbf{x}$ to define the integral in above formulae. Then, we will consider relations (71) and (72) as relations determining the invertible map from the operator $\hat{A}$ onto the function $f_{\hat{A}}(\mathbf{x})$. Multiplying both sides of Eq. (2) by the operator $\hat{U}(\mathbf{x}')$ and taking the

trace, one can satisfy the consistency condition for the operators $\hat{U}(\mathbf{x}')$ and $\hat{D}(\mathbf{x})$

$$\mathrm{Tr}\left[\hat{U}(\mathbf{x}')\hat{D}(\mathbf{x})\right] = \delta\left(\mathbf{x}' - \mathbf{x}\right). \tag{73}$$

The consistency condition (73) follows from the relation

$$f_{\hat{A}}(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{x}') f_{\hat{A}}(\mathbf{x}')\, d\mathbf{x}'. \tag{74}$$

The kernel in (74) is equal to the standard Dirac delta-function, if the set of functions $f_{\hat{A}}(\mathbf{x})$ is a complete set.

In fact, we could consider relations of the form

$$\hat{A} \to f_{\hat{A}}(\mathbf{x}) \tag{75}$$

and

$$f_{\hat{A}}(\mathbf{x}) \to \hat{A}. \tag{76}$$

The most important property of the map is the existence of the associative product (star-product) of the functions.

We introduce the product (star-product) of two functions $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$ corresponding to two operators $\hat{A}$ and $\hat{B}$ by the relationships

$$f_{\hat{A}\hat{B}}(\mathbf{x}) = f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) := \mathrm{Tr}\left[\hat{A}\hat{B}\hat{U}(\mathbf{x})\right]. \tag{77}$$

Since the standard product of operators on a Hilbert space is an associative product, i.e., $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, it is obvious that formula (77) defines an associative product for the functions $f_{\hat{A}}(\mathbf{x})$, i.e.,

$$f_{\hat{A}}(\mathbf{x}) * \left(f_{\hat{B}}(\mathbf{x}) * f_{\hat{C}}(\mathbf{x})\right) = \left(f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x})\right) * f_{\hat{C}}(\mathbf{x}). \tag{78}$$

Using formulae (71) and (72), one can write down a composition rule for two symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$, which determines the star-product of these symbols. The composition rule is described by the formula

$$f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) = \int f_{\hat{A}}(\mathbf{x}'') f_{\hat{B}}(\mathbf{x}') K(\mathbf{x}'', \mathbf{x}', \mathbf{x})\, d\mathbf{x}'\, d\mathbf{x}''. \tag{79}$$

The kernel in the integral of (79) is determined by the trace of the product of the basic operators, which we use to construct the map

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \mathrm{Tr}\left[\hat{D}(\mathbf{x}'')\hat{D}(\mathbf{x}')\hat{U}(\mathbf{x})\right]. \tag{80}$$

The kernel function satisfies the composition property $K * K = K$.

6. Tomographic representation

In this section, we will consider an example of the probability representation of quantum mechanics [42]. In the probability representation of quantum mechanics, the state is described by a family of probabilities [43–45]. According to the general scheme, one can introduce for the operator $\hat{A}$ the function $f_{\hat{A}}(\mathbf{x})$, where

$$\mathbf{x} = (x_1, x_2, x_3) \equiv (X, \mu, \nu),$$

which we denote here as $w_{\hat{A}}(X, \mu, \nu)$ depending on the position X and the parameters μ and ν of the reference frame

$$w_{\hat{A}}(X, \mu, \nu) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right]. \tag{81}$$

We call the function $w_{\hat{A}}(X, \mu, \nu)$ the tomographic symbol of the operator $\hat{A}$. The operator $\hat{U}(\mathbf{x})$ is given by

$$\begin{aligned}\hat{U}(\mathbf{x}) \equiv \hat{U}(X, \mu, \nu) \;&=\; \exp\left(\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right) \exp\left(\frac{i\theta}{2}\left(\hat{q}^2 + \hat{p}^2\right)\right) \mid X\rangle\langle X \mid \\ &\quad \times \exp\left(-\frac{i\theta}{2}\left(\hat{q}^2 + \hat{p}^2\right)\right) \exp\left(-\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right) \\ &=\; \hat{U}_{\mu\nu} \mid X\rangle\langle X \mid \hat{U}^{\dagger}_{\mu\nu}. \end{aligned} \tag{82}$$

The tomographic symbol is the homogeneous version of the Moyal phase-space density. The angle θ and parameter λ in terms of the reference phase-space frame parameters are given by

$$\mu = e^{\lambda}\cos\theta, \qquad \nu = e^{-\lambda}\sin\theta,$$

that is, $\hat{q}$ and $\hat{p}$ are position and momentum operators

$$\hat{q} \mid X\rangle = X \mid X\rangle \tag{83}$$

and $\mid X\rangle\langle X \mid$ is the projection density. One has the canonical transform of quadratures

$$\hat{X} = \hat{U}_{\mu\nu}\,\hat{q}\,\hat{U}^{\dagger}_{\mu\nu} = \mu\hat{q} + \nu\hat{p},$$

$$\hat{P} = \hat{U}_{\mu\nu}\,\hat{p}\,\hat{U}^{\dagger}_{\mu\nu} = \frac{1 + \sqrt{1 - 4\mu^2\nu^2}}{2\mu}\,\hat{p} - \frac{1 - \sqrt{1 - 4\mu^2\nu^2}}{2\nu}\,\hat{q}.$$

Using the approach of [46] one obtains the relationship

$$\hat{U}(X, \mu, \nu) = \delta(X - \mu\hat{q} - \nu\hat{p}).$$

In the case we are considering, the inverse transform determining the operator in terms of the tomogram [see Eq. (72)] will be of the form

$$\hat{A} = \int w_{\hat{A}}(X,\mu,\nu)\hat{D}(X,\mu,\nu)\,dX\,d\mu\,d\nu, \tag{84}$$

where

$$\hat{D}(\mathbf{x}) \equiv \hat{D}(X,\mu,\nu) = \frac{1}{2\pi}\exp\left(iX - i\nu\hat{p} - i\mu\hat{q}\right). \tag{85}$$

The trace of the above operator, which provides the kernel determining the trace of an arbitrary operator in the tomographic representation, reads

$$\mathrm{Tr}\,\hat{D}(\mathbf{x}) = e^{iX}\delta(\mu)\delta(\nu).$$

The function $w_{\hat{A}}(X,\mu,\nu)$ satisfies the relation

$$w_{\hat{A}}\left(\lambda X,\lambda\mu,\lambda\nu\right) = \frac{1}{|\lambda|}\,w_{\hat{A}}(X,\mu,\nu). \tag{86}$$

This means that the tomographic symbols of operators are homogeneous functions of three variables.

If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through the corresponding functions by the formulas

$$\begin{aligned}
\hat{A}_1 &= \int w_{\hat{A}_1}(X',\mu',\nu')\hat{D}(X',\mu',\nu')\,dX'\,d\mu'\,d\nu', \\
\hat{A}_2 &= \int w_{\hat{A}_2}(X'',\mu'',\nu'')\hat{D}(X'',\mu'',\nu'')dX''\,d\mu''\,d\nu'',
\end{aligned} \tag{87}$$

and $\hat{A}$ denotes the product of $\hat{A}_1$ and $\hat{A}_2$, then the function $w_{\hat{A}}(X,\mu,\nu)$, which corresponds to $\hat{A}$, is the star-product of the functions $w_{\hat{A}_1}(X,\mu,\nu)$ and $w_{\hat{A}_2}(X,\mu,\nu)$. Thus this product

$$w_{\hat{A}}(X,\mu,\nu) = w_{\hat{A}_1}(X,\mu,\nu) * w_{\hat{A}_2}(X,\mu,\nu)$$

reads

$$w_{\hat{A}}(X,\mu,\nu) = \int w_{\hat{A}_1}(\mathbf{x}'')w_{\hat{A}_2}(\mathbf{x}')K(\mathbf{x}'',\mathbf{x}',\mathbf{x})\,d\mathbf{x}''\,d\mathbf{x}', \tag{88}$$

with kernel given by

$$K(\mathbf{x}'',\mathbf{x}',\mathbf{x}) = \mathrm{Tr}\left[\hat{D}(X'',\mu'',\nu'')\hat{D}(X',\mu',\nu')\hat{U}(X,\mu,\nu)\right]. \tag{89}$$

The explicit form of the kernel reads

$$K(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X\mu,\nu)$$

$$= \frac{\delta\Big(\mu(\nu_1+\nu_2)-\nu(\mu_1+\mu_2)\Big)}{4\pi^2}\,\exp\left(\frac{i}{2}\Big\{(\nu_1\mu_2-\nu_2\mu_1)+2X_1+2X_2 - \left[\frac{1-\sqrt{1-4\mu^2\nu^2}}{\nu}\,(\nu_1+\nu_2)+\frac{1+\sqrt{1-4\nu^2\mu^2}}{\mu}\,(\mu_1+\mu_2)\right]X\Big\}\right). \tag{90}$$

7. Multipartite systems

Let us assume that for multimode (N-mode) system one has

$$\hat{U}(\vec{y}) = \prod_{k=1}^{N} \otimes \hat{U}\left(\vec{x}^{(k)}\right), \tag{91}$$

$$\hat{D}(\vec{y}) = \prod_{k=1}^{N} \otimes \hat{D}\left(\vec{x}^{(k)}\right), \tag{92}$$

where

$$\vec{y} = \Big(x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_m^{(N)}\Big). \tag{93}$$

This means that the symbol of the density operator of the composite system reads

$$f_\rho(\vec{y}) = \mathrm{Tr}\left[\hat{\rho}\prod_{k=1}^{N}\otimes\hat{U}(\vec{x}^{(k)})\right]. \tag{94}$$

The inverse transform reads

$$\hat{\rho} = \int d\vec{y}\, f_\rho(\vec{y}) \prod_{k=1}^{N}\otimes\hat{D}(\vec{x}^{(k)}), \qquad d\vec{y} = \prod_{k=1}^{N}\prod_{s=1}^{m} dx_s^{(k)}. \tag{95}$$

Now we formulate the properties of the symbols in the case of entangled and separable states, respectively.

Given a composite m-partite system with density operator $\hat{\rho}$.

If the nonnegative operator can be presented in the form of a "probabilistic sum"

$$\hat{\rho} = \sum_{\vec{z}} \mathcal{P}(\vec{z})\hat{\rho}_{\vec{z}}^{(a_1)} \otimes \hat{\rho}_{\vec{z}}^{(a_2)} \otimes \cdots \otimes \hat{\rho}_{\vec{z}}^{(a_m)}, \tag{96}$$

with positive probability distribution function $\mathcal{P}(\vec{z})$, where the components of $\vec{z}$ can be either discrete or continuous, we call the state a "separable state." Without loss of generality, all factors in the tensor

products can be considered as projectors of rank one. This means that the symbol of the state can be presented in the form

$$f_\rho(\vec{y}) = \sum_{\vec{z}} \mathcal{P}(\vec{z}) \prod_{k=1}^{m} f_\rho^{(a_k)}(\vec{x}_k, \vec{z}). \tag{97}$$

Analogous formula can be written for the tomogram of separable state. We point out that in the multipartite case one can introduce random symbols and represent the symbol of separable density matrix of composite system as mean value of pointwise products of symbols of subsystem density operators. As in the bipartite case, one can use sum over different indices but this sum can be always reduced to the sum over only one index common for all the subsystems. It is important that for separable state its symbol always can be represented as the sum containing number of summants which is equal to dimensionality of composite system. Each term in the sum is equal to mean value of random projector. The random projector is constructed as the product of diagonal projectors of rank one in each subsystem considered in random local basis obtained by means of random unitary local transforms.

8. Spin tomography

Below we concentrate on bipartite spin systems.

The tomographic probability (spin tomogram) completely determines the density matrix of a spin state. It has been introduced in [31, 32, 36]. The tomographic probability for the spin-j state is defined via the density matrix by the formula

$$\langle jm \mid D^\dagger(g)\rho D(g) \mid jm\rangle = W^{(j)}(m, \vec{0}), \qquad m = -j, -j+1, \ldots, j, \tag{98}$$

where $D(g)$ is the matrix of $SU(2)$-group representation depending on the group element g determined by three Euler angles. It is useful to generalize the construction of spin tomogram.

One can introduce unitary spin tomograms $w(m, u)$ by replacing in above formula (98) the matrix $D(g)$ by generic unitary matrix u. For the case of higher spins $j = 1, 3/2, 2, \ldots$, the $n\times n$ projector matrix

$$\rho_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad n = 2j+1 \tag{99}$$

has the unitary spin tomogram denoted as

$$w_1(j,u) = |u_{11}|^2, \quad w_1(j-1,u) = |u_{12}|^2, \quad \dots \quad w_1(-j,u) = |u_{1n}|^2. \tag{100}$$

Other projectors

$$\rho_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & 0 \end{pmatrix}, \tag{101}$$

in which unity is located in kth column, have the tomogram $w_k(m,u)$ of the form

$$w_k(j,u) = |u_{k1}|^2, \quad w_k(j-1,u) = |u_{k2}|^2, \quad \dots \quad w_k(-j,u) = |u_{kn}|^2. \tag{102}$$

In connection with the decomposition of any density matrix in the form

$$\rho = \sum_{jk} \rho_{jk} E_{jk}, \tag{103}$$

the unitary spin tomogram can be presented in form of the decomposition

$$w_\rho(m,u) = \sum_{jk} \rho_{jk} w_{jk}(m,u), \tag{104}$$

where $w_{jk}(m,u)$ are basic unitary spin symbols of transition operators E_{jk} of the form

$$w_{jk}(m,u) = \langle jm \mid u^\dagger E_{jk} u \mid jm \rangle. \tag{105}$$

If one uses a map

$$\rho \to \rho', \tag{106}$$

the unitary spin tomogram is transformed as

$$w_\rho(m,u) \to w'_\rho(m,u) = \sum_{jk} \rho'_{jk} w_{jk}(m,u). \tag{107}$$

If the transform (106) is a linear one

$$\rho_{jk} \to \rho'_{jk} = L_{jk,ps} \rho_{ps}, \tag{108}$$

the transform reads

$$w'_\rho(m,u) = \sum_{ps} \rho_{ps} w'_{ps}(m,u). \tag{109}$$

Here

$$w'_{ps}(m,u) = \sum_{jk} L_{jk,ps} w_{jk}(m,u) \tag{110}$$

is the linear transform of the basic tomographic symbols of the operators E_{jk}.

Let us now discuss some properties of usual spin tomograms.

The set of the tomogram values for each $\vec{0}$ is an overcomplete set. We need only a finite number of independent locations which will give information on the density matrix of the spin state. Due to the structure of the formula, there are only two Euler angles involved. They are combined into the unit vector

$$\vec{0} = (\cos\phi \sin\vartheta, \sin\phi \sin\vartheta, \cos\vartheta). \tag{111}$$

This is the Hopf map from S^3 to S^2.

The physical meaning of the probability $W(m,\vec{0})$ is the following.

It is the probability to find, in the state with the density matrix ρ, the spin projection on direction $\vec{0}$ equal to m. For a bipartite system, the spin tomogram is defined as follows:

$$W(m_1 m_2 \vec{0}_1 \vec{0}_2) = \langle j_1 m_1 j_2 m_2 \mid D^\dagger(g_1) D^\dagger(g_2) \rho D(g_1) D(g_2) \mid j_1 m_1 j_2 m_2 \rangle. \tag{112}$$

It completely determines the density matrix ρ. It has the meaning of the joint probability distribution for spin j_1 and j_2 projections m_1 and m_2 on directions $\vec{0}_1$ and $\vec{0}_2$. Since the map $\rho \rightleftharpoons W$ is linear and invertible, the definition of separable system can be rewritten in the following form for the decomposition of the joint probability into a sum of products (of factorized probabilities):

$$W(m_1 m_2 \vec{0}_1 \vec{0}_2) = \sum_k p_k W^{(k)}(m_1 \vec{0}_1) \tilde{W}^{(k)}(m_2 \vec{0}_2). \tag{113}$$

This form can be considered to formulate the criterion of separability of the two-spin state.

One can present this formula in the form

$$W(m_1 m_2 \vec{0}_1 \vec{0}_2) = \langle W(m_1 \vec{0}_1) \tilde{W}(m_2 \vec{0}_2) \rangle, \tag{114}$$

where we interpret the positive numbers p_k as probability distributions. Thus separability means the possibility to represent joint probability distribution in the form of average product of two random probability distributions.

The state is separable iff the tomogram can be written in the form (113) with $\sum_k p_k = 1$, $p_k \geq 0$. It seems that we simply use the definition but, in fact, we cast the problem of separability into the form of the

property of the positive joint probability distribution of two random variables. This is an area of probability theory and one can use the results and theorems on joint probability distributions. If one does not use any theorem, one has to study the solvability of relation (113) considered as the equation for unknown probability distribution p_k and unknown probability functions $W^{(k)}(m_1\vec{0}_1)$ and $W^{(k)}(m_2\vec{0}_2)$.

9. Example of spin-1/2 bipartite system

For the spin-1/2 state, the generic density matrix can be presented in the form

$$\rho = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\vec{n}\right), \qquad \vec{n} = (n_1, n_2, n_3), \tag{115}$$

where $\vec{\sigma}$ are Pauli matrices and $\vec{n}^2 \leq 1$, with the vector $\vec{n}$ for a pure state being the unit vector. This decomposition means that we use as basis in 4-dimensional vector space the vectors corresponding to the Pauli matrices and the unit matrix, i.e.,

$$\vec{\sigma}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{\sigma}_2 = \begin{pmatrix} 0 \\ -i \\ i \\ 0 \end{pmatrix}, \qquad \vec{\sigma}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{116}$$

The density matrix vector

$$\vec{\rho} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} \tag{117}$$

is decomposed in terms of the basis vectors

$$\vec{\rho} = \frac{1}{2}\left(\vec{1} + n_1\vec{\sigma}_1 + n_2\vec{\sigma}_2 + n_3\vec{\sigma}_3\right). \tag{118}$$

This means that the spin tomogram of the spin-1/2 state can be given in the form

$$W\left(\frac{1}{2}, \vec{0}\right) = \frac{1}{2} + \frac{\vec{n}\cdot\vec{0}}{2}, \qquad W\left(-\frac{1}{2}, \vec{0}\right) = \frac{1}{2} - \frac{\vec{n}\cdot\vec{0}}{2}. \tag{119}$$

We can consider tomograms of specific spin state. If the state is pure state with density matrix

$$\rho_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{120}$$

the spin tomogram $W(m,\vec{0})$, where

$$m=\pm\frac{1}{2},\qquad \vec{0}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$$

has the values

$$W_+\left(\frac{1}{2},\vec{0}\right)=\cos^2\frac{\theta}{2},\quad W_+\left(-\frac{1}{2},\vec{0}\right)=\sin^2\frac{\theta}{2}. \tag{121}$$

The tomogram of the pure state

$$\rho_-=\begin{pmatrix}0&0\\0&1\end{pmatrix}, \tag{122}$$

has the values

$$\begin{aligned} W_-\left(\frac{1}{2},\vec{0}\right) &= \sin^2\frac{\theta}{2}=\cos^2\frac{\pi-\theta}{2},\\ W_-\left(-\frac{1}{2},\vec{0}\right) &= \cos^2\frac{\theta}{2}=\sin^2\frac{\pi-\theta}{2}. \end{aligned} \tag{123}$$

The spin tomogram of the diagonal density matrix

$$\rho_d=\begin{pmatrix}\rho_{11}&0\\0&\rho_{22}\end{pmatrix} \tag{124}$$

equals

$$W_d(m,\vec{0})=\rho_{11}W_+(m,\vec{0}_+)+\rho_{22}W_-(m,\vec{0}_-). \tag{125}$$

The generic density matrix which has eigenvalues ρ_{11} and ρ_{22} can be presented in the form

$$u_0\rho_d u_0^\dagger, \tag{126}$$

where the unitary matrix u_0 has columns containing components of normalized eigenvectors of the density matrix ρ.

This means that the tomogram of the state with the matrix ρ reads

$$W_\rho(m,\vec{0})=\langle m\mid u^\dagger u_0\rho_d u_0^\dagger u\mid m\rangle. \tag{127}$$

The elements of the group can be combined

$$u^\dagger u_0=\tilde{u}. \tag{128}$$

Thus the tomogram becomes

$$W_\rho(m,\vec{0})=W_d(m,\vec{0}'), \tag{129}$$

where the angle $\vec{0}'$ corresponds to the Euler angle calculated from the product of two unitary matrices $u_0^\dagger u$.

One can use the property of numbers ρ_{11} and ρ_{22} to interpret formula (125) as averaging

$$W_d(m,\vec{0}) = \langle W(m,\vec{0}')\rangle, \tag{130}$$

where one interprets two functions $W_+(m,\vec{0})$ and $W_-(m,\vec{0}')$ as the realization of "random" probability distribution function $W_\pm(m,\vec{0})$. Then one has

$$W_\rho(m,\vec{0}) = \langle W(m,\vec{0}')\rangle. \tag{131}$$

The new vector $\vec{0}'$ has the parameter θ' obtained from the initial parameter θ by action of the unitary matrix on the initial unitary matrix u.

Inserting these probability values into relation (113) for each value of k, we get the relationships

$$\begin{aligned} W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right) &= \frac{1}{4}+\frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1+\frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 \\ &\quad +\sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^*\cdot\vec{0}_2\right), \end{aligned} \tag{132}$$

$$\begin{aligned} W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right) &= \frac{1}{4}+\frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1-\frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 \\ &\quad -\sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^*\cdot\vec{0}_2\right), \end{aligned} \tag{133}$$

$$\begin{aligned} W\left(-\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right) &= \frac{1}{4}-\frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1+\frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 \\ &\quad -\sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^*\cdot\vec{0}_2\right). \end{aligned} \tag{134}$$

One has the normalization property

$$\sum_{m_1,\,m_2=-1/2}^{1/2} W(m_1m_2\vec{0}_1\vec{0}_2) = 1. \tag{135}$$

One easily gets

$$W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right) = \frac{1}{2}+\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1. \tag{136}$$

This means that the derivative in $\vec{0}_1$ on the left-hand side gives

$$\frac{\partial}{\partial\vec{0}_1}\left[W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]=\left(\sum_k p_k\vec{n}_k\right). \quad (137)$$

Analogously

$$\frac{\partial}{\partial\vec{0}_2}\left[W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(-\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]=\left(\sum_k p_k\vec{n}_k^{(\star)}\right). \quad (138)$$

Taking the sum of (133) and (134)) one sees that

$$\frac{1}{2}\frac{\partial}{\partial\vec{0}_i}\frac{\partial}{\partial\vec{0}_j}\left[W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(-\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]$$
$$=-\sum_k p_k(n_k)_i(n_k^{(\star)})_j. \quad (139)$$

Since we look for the solution where $p_k\geq 0$, we can introduce

$$\vec{N}_k=\sqrt{p_k}\vec{n}_k,\qquad \vec{N}_k^{(\star)}=\sqrt{p_k}\vec{n}_k^{(\star)}. \quad (140)$$

This means that the derivative in (139) can be presented as a tensor

$$-T_{ij}=\sum_k(N_k)_i(N_k^{(\star)})_j. \quad (141)$$

One has

$$\sum_k p_k\vec{n}_k=\sum_k\sqrt{p_k}\vec{N}_k, \quad (142)$$

$$\sum_k p_k\vec{n}_k^{\star}=\sum_k\sqrt{p_k}\vec{N}_k^{(\star)}. \quad (143)$$

The conditions of solvability of the obtained equations is a criterion for separability or entanglement of a bipartite quantum spin state. Using the arguments on the representation of the tomogram (tomographic symbol) as sum of random basic projector symbols we get that for two qubits the separable state has the tomogram with following properties. All four values of joint probability distribution function are equal to mean values of product of two cosine of two different angles squared, product of sine of two different angles squared and product of sine and cosine squared, respectively. The entangled matrix does not provide such structure.

As an example, we consider the Werner state. For the Werner state (see, e.g., [47]) with the density matrix

$$\rho_{AB}=\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix},$$

(144)

$$\rho_A = \rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

one can reconstruct the known results that for $p < 1/3$ the state is separable and for $p > 1/3$ the state is entangled, since in the decomposition of the density operator in the form (113) the state

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{145}$$

has the weight $p_0 = (1 - 3p)/4$.

For $p > 1/3$, the coefficient p_o becomes negative.

There is some extension of the presented consideration.

Let us consider the state with the density matrix (nonnegative and Hermitian)

$$\rho = \begin{pmatrix} R_{11} & 0 & 0 & R_{12} \\ 0 & \rho_{11} & \rho_{12} & 0 \\ 0 & \rho_{21} & \rho_{22} & 0 \\ R_{21} & 0 & 0 & R_{22} \end{pmatrix}, \qquad \mathrm{Tr}\,\rho = 1. \tag{146}$$

Using the procedure of mapping the matrix onto vector $\vec{\rho}$ and applying to the vector the nonlocal linear transform corresponding to the Peres partial transpose and making the inverse map of the transformed vector onto the matrix, we obtain

$$\rho^m = \begin{pmatrix} R_{11} & 0 & 0 & \rho_{12} \\ 0 & \rho_{11} & R_{12} & 0 \\ 0 & R_{21} & \rho_{22} & 0 \\ \rho_{21} & 0 & 0 & R_{22} \end{pmatrix}. \tag{147}$$

In the case of separable matrix ρ, the matrix ρ^m is a nonnegative matrix. Calculating the eigenvalues of ρ^m and applying the condition of their positivity, we get

$$R_{11}R_{22} \geq |\rho_{12}|^2, \qquad \rho_{11}\rho_{22} \geq |R_{12}|^2. \tag{148}$$

Violation of these inequalities gives a signal that ρ is entangled. For Werner state (144), Eq. (148) means

$$1 + p > 0, \qquad 1 - p > 2p, \tag{149}$$

which recovers the condition of separability $p < 1/3$ mentioned above.

The joint probability distribution (112) of separable state is positive after making the local and nonlocal (partial transpose-like) transforms connected with positive map semigroup. But for entangled state, function (112) can take negative values after making this map in the function and replacing on the right-hand side of this equality the product of two matrices $D(g)$ by generic unitary transform u. This is a criterion of entanglement in terms of unitary spin tomogram of the state of multiparticle system.

A simpler and more transparent case is the generalized Werner model with density matrix

$$\rho = \frac{1}{4}\left(1 + \mu_1\sigma_1 \otimes \tau_1 + \mu_2\sigma_2 \otimes \tau_2 + \mu_3\sigma_3 \otimes \tau_3\right). \tag{150}$$

Here the density matrix is expressed in terms of tensor products of two sets of Pauli matrices σ_k and τ_k $(k = 1, 2, 3)$, which are chosen in the standard form.

Its eigenvalues are

$$1-\mu_1-\mu_2-\mu_3, \quad 1+\mu_1+\mu_2-\mu_3, \quad 1+\mu_1-\mu_2+\mu_3, \quad 1-\mu_1+\mu_2+\mu_3.$$

These eigenvalues are related to the vertices of a regular tetrahedron. The partially time-reversed density matrix is

$$\widetilde{\rho} = \frac{1}{4}\left(1 - \mu_1\sigma_1 \otimes \tau_1 - \mu_2\sigma_2 \otimes \tau_2 - \mu_3\sigma_3 \otimes \tau_3\right), \tag{151}$$

which may be viewed as

$$\widetilde{\rho} = L^{(1)} \otimes L^{(2)}\rho \qquad \text{with} \qquad L^{(1)}\rho^{(1)} = \rho^{(1)}, \quad L^{(2)}\rho^{(2)} = 1 - \rho^{(2)}. \tag{152}$$

The eigenvalues of this are

$$1+\mu_1+\mu_2+\mu_3, \quad 1+\mu_1-\mu_2-\mu_3, \quad 1-\mu_1+\mu_2-\mu_3, \quad 1-\mu_1-\mu_2+\mu_3.$$

These form an inverted tetrahedron and they have the common domain which is a regular octahedron. The unitary spin tomograms can be written down by inspection and we may verify that all the relations required by the separability criterion (see the next section for details) are satisfied by any point inside the octahedron for ρ and for $L^{(1)}\otimes L^{(2)}\rho$ but the relations connected with positivity condition expressed in terms of positivity of unitary spin tomogram fail when it lies outside.

10. Tomogram of the group $U(n)$

In this section we discuss in more detail the separability criterion using introduced notion of unitary spin tomogram.

In order to formulate a criterion of separability for a bipartite spin system with spin j_1 and j_2, we introduce the tomogram $w(\vec{l},\vec{m},g^{(n)})$ for the group $U(n)$, where

$$n = n_1 n_2, \qquad n_1 = 2j_1 + 1, \qquad n_2 = 2j_2 + 1,$$

and $g^{(n)}$ are parameters of the group element. Vectors $\vec{l}$ and $\vec{m}$ label a basis $\mid \vec{l},\vec{m}\rangle$ of the fundamental representation of the group $U(n)$. For example, since this representation is irreducible, being reduced to the representation of the $U(n_1)\otimes U(n_2)$ subgroup of the group $U(n)$, the basis can be chosen as the product of basis vectors:

$$\mid j_1,m_1\rangle \mid j_2,m_2\rangle = \mid j_1,j_2,m_1,m_2\rangle. \tag{153}$$

Due to the irreducibility of this representation of the group $U(n)$ and its subgroup, there exists a unitary transform $u^{\vec{l}\vec{m}}_{j_1j_2m_1m_2} \mid \vec{l},\vec{m}\rangle$ such that

$$\mid j_1,j_2,m_1,m_2\rangle = \sum_{\vec{l}\vec{m}} u^{\vec{l}\vec{m}}_{j_1j_2m_1m_2} \mid \vec{l},\vec{m}\rangle, \tag{154}$$

$$\mid \vec{l}\vec{m}\rangle = \sum_{m_1m_2} (u^{-1})^{\vec{l}\vec{m}}_{j_1j_2m_1m_2} \mid j_l,j_2,m_1,m_2\rangle. \tag{155}$$

One can define the $U(n)$ tomogram for a Hermitian nonnegative $n\times n$ density matrix ρ, which belongs to the Lie algebra of the group $U(n)$, by a generic formula

$$w(\vec{l},\vec{m},g^{(n)}) = \langle \vec{l},\vec{m} \mid U^\dagger(g^{(n)})\rho U(g^{(n)}) \mid \vec{l},\vec{m}\rangle. \tag{156}$$

Formula (156) defines the tomogram in the basis $\mid \vec{l},\vec{m}\rangle$ for arbitrary irreducible representation of the unitary group. But below we focus only on tomograms connected with spins.

Let us define the $U(n)$ tomogram using the basis $\mid j_1,j_2,m_1,m_2\rangle$ namely for fundamental representation, i.e.,

$$\begin{aligned} &w^{(j_1,j_2)}(m_1,m_2,g^{(n)}) \\ &= \langle j_1,j_2,m_1,m_2 \mid U^\dagger(g^{(n)})\rho U(g^{(n)}) \mid j_1,j_2,m_1,m_2\rangle. \end{aligned} \tag{157}$$

This unitary spin tomogram becomes the spin-tomogram [34] for the $g^{(n)} \in U(2)\otimes U(2)$ subgroup of the group $U(n)$. The properties of this tomogram follow from its definition as the joint probability distribution of two random spin projections m_1, m_2 depending on $g^{(n)}$ parameters.

One has the normalization condition

$$\sum_{m_1,m_2} w^{(j_1,j_2)}(m_1,m_2,g^{(n)}) = 1. \tag{158}$$

Also all the probabilities are nonnegative, i.e.,

$$w^{(j_1,j_2)}(m_l,m_2,g^{(n)}) \geq 0. \tag{159}$$

Due to this, one has

$$\sum_{m_1,m_2} |w^{(j_1,j_2)}(m_l,m_2,g^{(n)})| = 1. \tag{160}$$

For the spin-tomogram,

$$g^{(n)} \to \left(\vec{O}_1, \vec{O}_2\right) \tag{161}$$

and

$$w^{(j_1,j_2)}(m_l,m_2,g^{(n)}) \to w(m_1,m_2,\vec{O}_1,\vec{O}_2). \tag{162}$$

The separability and entanglement condition discussed in the previous section for a bipartite spin-tomogram can be considered also from the viewpoint of the properties of a $U(n)$ tomogram. If the two-spin $n\times n$ density matrix ρ is separable, it remains separable under the action of the generic positive map of the subsystem density matrices. This map can be described as follows.

Let ρ be mapped onto vector $\vec{\rho}$ with n^2 components. The components are simply ordered rows of the matrix ρ, i.e.,

$$\vec{\rho} = \Big(\rho_{11},\rho_{12},\ldots,\rho_{1n},\rho_{21},\rho_{22},\ldots,\rho_{nn},\Big). \tag{163}$$

Let the $n^2\times n^2$ matrix L be taken in the form

$$L = \sum_s p_s L_s^{(j_1)} \otimes L_s^{(j_2)}, \qquad p_s \geq 0, \quad \sum_s p_s = 1, \tag{164}$$

where the $n_1\times n_1$ matrix $L_s^{(j_1)}$ and the $n_2\times n_2$ matrix $L_s^{(j_2)}$ describe the positive maps of density matrices of spin-j_1 and spin-j_2 subsystems, respectively. We map vector $\vec{\rho}$ onto vector $\vec{\rho}_L$

$$\vec{\rho}_L = L\vec{\rho} \tag{165}$$

and construct the $n\times n$ matrix ρ_L, which corresponds to the vector $\vec{\rho}_L$. Then we consider the $U(n)$ tomogram of the matrix ρ_L, i.e.,

$$\begin{aligned} &w_L^{(j_1,j_2)}(m_l,m_2,g^{(n)}) \\ &= \langle j_1,j_2,m_1,m_2 \mid U^\dagger(g^{(n)})\rho_L U(g^{(n)}) \mid j_1,j_2,m_l,m_2\rangle. \end{aligned} \tag{166}$$

Using this tomogram we introduce the function

$$F(g^{(n)}, L) = \sum_{m_1,m_2} \left| w_L^{(j_1,j_2)}(m_1, m_2, g^{(n)}) \right|. \tag{167}$$

For separable states, this function does not depend on the $U(n)$-group parameter $g^{(n)}$ and positive-map matrix elements of the matrix L.

For the normalized density matrix ρ of the bipartite spin-system, this function reads

$$F(g^{(n)}, L) = 1. \tag{168}$$

For entangled states, this function depends on $g^{(n)}$ and L and is not equal to unity. This property can be chosen as a necessary and sufficient condition for separability of bipartite spin-states. We introduce also tomographic purity parameter μ_k of kth order by the formula

$$\mu_k(g^{(n)}, L) = \sum_{m_1 m_2} \left| w_L^{(j_1,j_2)}(m_1, m_2, g^{(h)}) \right|^k.$$

For identity semigroup element L and specific $g_0^{(n)}$ unitary transform diagonalizing the density matrix, the tomographic purity μ_2 is identical to purity parameter of the state ρ. The parameters for $k = 2, 3, \ldots$, correspond to $\operatorname{Tr} \rho^{k+1}$.

In fact, the formulated approach can be extended to multipartite systems too. The generalization is as follows.

Given N spin-systems with spins $j_1, j_2, \ldots, j_N$, let us consider the group $U(n)$ with

$$n = \prod_{k=1}^{N} n_k, \qquad n_k = 2j_k + 1. \tag{169}$$

Let us introduce the basis

$$\mid \vec{m}\rangle = \prod_{k=1}^{N} \mid j_k m_k\rangle \tag{170}$$

in the linear space of the fundamental representation of the group $U(n)$. We define now the $U(n)$ tomogram of a state with the $n \times n$ matrix ρ:

$$w_\rho(\vec{m}, g^{(n)}) = \langle \vec{m} \mid U^\dagger(g^{(n)}) \rho U(g^{(n)}) \mid \vec{m}\rangle. \tag{171}$$

For a positive Hermitian matrix ρ with $\operatorname{Tr} \rho = 1$, we formulate the criterion of separability as follows.

Let the map matrix L be of the form

$$L = \sum_s p_s \Big(\prod_{k=1}^{N} \otimes L_s^{(k)} \Big), \qquad p_s \geq 0, \quad \sum_s p_s = 1, \tag{172}$$

where $L_s^{(k)}$ is the positive-map matrix of the density matrix of the kth spin subsystem. We construct the matrix ρ_L as in the case of the bipartite system using the matrix L. The function

$$F(g^{(n)}, L) = \sum_{\vec{m}} |w_{\rho_L}(\vec{m}, g^{(n)})| \geq 1 \tag{173}$$

is equal to unity for separable state and depends on the matrix L and $U(n)$-parameters $g^{(n)}$ for entangled states.

This criterion can be applied also in the case of continuous variables, e.g., for Gaussian states of photons. Function (173) can provide the measure of entanglement. Thus one can use the maximum value (or a mean value) of this function as a characteristic of entanglement. In the previous section, we considered the generalized Werner states. Using the above criterion, one can get the domain of values of the parameters of the states for which one has separability or entanglement. In fact, the separability criterion is related to the following positivity criterion of finite or infinite (trace class) matrix A. The matrix A is positive iff the sum of moduli of diagonal matrix elements of the matrix $UAU^\dagger$ is equal to a positive trace of the matrix A for an arbitrary unitary matrix U.

11. Dynamical map and purification

In this section, we consider the connection of positive maps with purification procedure. In fact, formula

$$\rho \to \rho' = \sum_k p_k U_k \rho U_k^\dagger, \tag{174}$$

where U_k are unitary operators, can be considered in the form

$$\rho \to \rho' = \sum_k p_k \rho_k, \qquad p_k \geq 0, \qquad \sum_k p_k = 1. \tag{175}$$

Here the density operators ρ_k read

$$\rho_k = U_k \rho U_k^\dagger, \tag{176}$$

and the maps which are not sufficiently general keep the most degenerate density matrix fixed. This form is the form of probabilistic addition. This mixture of density operators can be purified with the help of a fiducial rank one projector P_0

$$\rho' \to \rho'' = N \left[\sum_{kj} \sqrt{p_k p_j}\, \frac{\rho_k P_0 \rho_j}{\sqrt{\mathrm{Tr}\, \rho_k P_0 \rho_j P_0}} \right], \tag{177}$$

where N is a normalization constant

$$N^{-1} = \mathrm{Tr}\left(\sum_{kj}\sqrt{p_k p_j}\,\frac{\rho_k P_0 \rho_j}{\sqrt{\mathrm{Tr}\,\rho_k P_0 \rho_j P_0}}\right). \tag{178}$$

The normalization is unnecessary if all ρ_k are mutually orthogonal. We call this map a purification map. It maps the density matrix of mixed state on the density matrix of pure state.

The map (174) could be interpreted as the evolution in time of the initial matrix ρ_0 considering unitary operators $U_k(t)$ depending on time. Thus one has

$$\rho_0 \to \rho(t) = \sum_k p_k U_k(t)\rho_0 U_k^\dagger(t). \tag{179}$$

In this case, the purification procedure provides the dynamical map of a pure state

$$|\,\psi_0\rangle\langle\psi_0\,| \to |\,\psi(t)\rangle\langle\psi(t)\,|, \tag{180}$$

where $|\,\psi(t)\rangle$ obeys a nonlinear equation and, in the general case, this map does not define a one parameter group of transformations not even locally.

For some specific cases, the evolution (179) can be described by a semigroup. The density matrix (179) obeys then a first-order differential equation in time for this case [27–29].

More specifically, the reason why there is no differential equation in time for the generic case is due to the absence of the property

$$\rho_{ij}(t_2) = \sum_{mn} K_{ij}^{mn}(t_2, t_1)\rho_{mn}(t_1), \tag{181}$$

where the kernel of evolution operator satisfies

$$K_{ij}^{mn}(t_3, t_2)K_{mn}^{pq}(t_2, t_1) = K_{ij}^{pg}(t_3, t_1). \tag{182}$$

Thus, via a purification procedure and a dynamical map applied to a density matrix we get a pure state (nonlinear dynamical map). This map can be used in nonlinear models of quantum evolution. Many linear positive maps both completely positive and not completely positive are contractive. We define a positive map L as "contractive" or "dilating" if $\mathrm{Tr}\,(L\rho)^2 \le \mathrm{Tr}\,(\rho)^2$ or $\mathrm{Tr}\,(L\rho)^2 \ge \mathrm{Tr}\,(\rho)^2$, respectively. This means, for example, that purity parameter $\mu = \mathrm{Tr}\,\rho^2$ after performing the positive map generically becomes smaller. There are maps for which the purity parameter is preserved, for example,

$$\rho \to \rho^{\mathrm{tr}}, \qquad \rho \to -\rho + \frac{2}{N}\,1. \tag{183}$$

These linear maps include also unitary transform

$$\rho \to u\rho u^\dagger. \tag{184}$$

There are maps which provide dilation. For qubit system, matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{185}$$

acting on arbitrary vector $\vec{\rho}_0$ corresponding to a density matrix ρ_0 gives the pure state

$$\rho_f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{186}$$

The matrices L_ε the inverse matrices exist for $\varepsilon \neq 0$. But these inverse matrices do not provide positive trace preserving maps. Since the purification procedure provides a positive map, which increases the purity parameter, the composition of linear map with the purification map provides the possibility to recover the initial density matrix ρ which was the object of action of a positive linear map. It means that the purification map $\hat{L}_p$ can give

$$\hat{L}_{P_0}(L\rho) = 1\rho \tag{187}$$

for any density matrix ρ but the choice of fiducial projector depends on ρ (the initial condition).

Thus one has also for completely positive maps

$$\rho \to \rho' = \sum_k \rho'_k, \qquad \rho_k = V_k \rho V_k^\dagger, \qquad \sum_k V_k^\dagger V_k = 1. \tag{188}$$

Making polar decomposition

$$\rho_k = \sqrt{\rho_{0k}} U_k, \qquad U_k U_k^\dagger = 1, \qquad \rho_{0k} \geq 0$$

and introducing the positive numbers $p_k = \operatorname{Tr} \rho_{0k}$, we construct the map

$$\rho' \to \rho'' = \left\{ \sum_{kj} \sqrt{p_k p_j} \frac{\tilde{\rho}_k P_0 \tilde{\rho}_j + \tilde{\rho}_j P_0 \tilde{\rho}_k}{\sqrt{T \tilde{\rho}_k P_0 \tilde{\rho}_j P_0}} \right\}, \quad \sum_k p_k = 1, \quad p_k \tilde{\rho}_k = \rho_k. \tag{189}$$

The matrix ρ'' is a matrix of rank one for any rank one fiducial projector P_0. The projector P_0 is restricted to be not orthogonal to the generic matrix ρ_k. Taking N orthogonal projectors $P_0^{(s)}$ $(s = 1, 2, \ldots, N)$ and

obtaining N projectors ρ_s'', one can combine them in order to get the initial matrix ρ. It means that one can take convex sum of the N pure states ρ_s'' to recover the initial mixed state ρ. Another way to make the state with higher purity was demonstrated using the modified purification procedure in [48]. For qubit state, one has

$$\rho = p_1\rho_1 + p_2\rho_2 + \kappa\sqrt{p_1p_2}\,\frac{\rho_1P_0\rho_2 + \rho_2P_0\rho_1}{\sqrt{\mathrm{Tr}\,\rho_1P_0\rho_2P_0}}\,, \quad p_1 + p_2 = 1, \tag{190}$$

where the decoherence parameter $0 \leq \kappa \leq 1$ is used. If $\kappa \sim 1$, we increase purity.

Let us discuss the map (188) using its matrix form, i.e.,

$$\rho_{\alpha\beta} \to \rho'_{\alpha\beta} = \sum_{ij} \mathcal{L}_{\alpha\beta,ij}\rho_{ij}. \tag{191}$$

The matrix $\mathcal{L}_{\alpha\beta,ij}$ is expressed in terms of the matrices V_k as

$$\mathcal{L}_{\alpha\beta,ij} = \sum_k (V_k)_{\alpha i}(V_k^*)_{\beta j}. \tag{192}$$

One can construct another positive map [49]

$$\rho \to \rho' = \sum_k r_k \mathrm{Tr}\,(R_k\rho)\,, \tag{193}$$

where r_k are density matrices and R_k are positive operators satisfying the normalization condition

$$\sum_k R_k = 1. \tag{194}$$

The matrix corresponding to this map (called entanglement breaking map [50]) reads

$$\mathcal{L}^b_{\alpha\beta,ij} = \sum_k (r_k)_{\alpha\beta}(R_k^*)_{ij}. \tag{195}$$

The entanglement breaking map is contractive positive map. There exist some special cases of completely positive maps. For example,

$$\rho \to -\varepsilon\rho + \frac{1+\varepsilon}{N}\,\rho \tag{196}$$

differs from the depolarizing map by replacing the unity operator by the density operator. Another map reads

$$\rho \to \frac{1 - \mathrm{diag}\,\rho}{N}\,. \tag{197}$$

The decoherence map (phase damping map) of the kind

$$\rho_{ij} \to \begin{cases} \rho_{ij}, & i = j \\ \lambda\rho_{ij}, & i \neq j, \end{cases} \tag{198}$$

where $|\lambda| < 1$ provides contractive map with uniform change of off-diagonal matrix elements of the density matrix.

Let us discuss the property of tomogram of bipartite system with density matrix ρ_{12}. If the density matrix is separable, than the depolarizing map of the second subsystem provides the following density matrix

$$\rho_{12} \to \rho_\varepsilon = -\varepsilon\rho_{12} + \frac{1+\varepsilon}{N_2}\,\underline{\rho^{(1)}} \otimes 1_2, \tag{199}$$

where

$$\underline{\rho^{(1)}} = \mathrm{Tr}_2(\rho_{12}) \tag{200}$$

and 1_2 is the N_2-dimensional unity matrix. Then one has the property of unitary spin tomogram

$$w_\varepsilon(m_1, m_2, g^{(n)}) = -\varepsilon w_{12}(m_1, m_2, g^{(n)}) + \frac{1+\varepsilon}{N_2}\,\underline{w}(m_1, m_2, g^{(n)}), \tag{201}$$

where $g^{(n)}$ is matrix of $U\Big((2j_1+1)(2j_2+1)\Big)$ unitary transform;
$w_\varepsilon(m_1, m_2, g^{(n)})$ is the tomogram of transformed density matrix of bipartite system;
$\underline{w}(m_1, m_2, g^{(n)})$ is the unitary spin tomogram of tensor product of partial trace $\underline{\rho^{(1)}}$ over the second subsystem's coordinates of the density matrix ρ_{12} and unity operator 1_2;
$w_{12}(m_1, m_2, g^{(n)})$ is the unitary spin tomogram of the state with density matrix ρ_{12}.

The criterion of separability means

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \left| \frac{1+\varepsilon}{2j_2+1} \underline{w}(m_1, m_2, g^{(n)}) - \varepsilon w_{12}(m_1, m_2, g^{(n)}) \right| = 1 \tag{202}$$

for arbitrary $g^{(n)}$ and ε.

For Werner states ρ_W, the tomogram of transformed state (in this case, it means that $p \to -\varepsilon p$) is related to the initial-state tomogram w_W

$$w_\varepsilon(m_1, m_2, g^{(n)}) = -\varepsilon w_{12}(m_1, m_2, g^{(n)}) + \frac{1+\varepsilon}{4}\,. \tag{203}$$

The criterion of separability yields

$$\sum_{m_1, m_2=-1/2}^{1/2} \left| \frac{1+\varepsilon}{4} - \varepsilon w_W(m_1, m_2, g^{(n)}) \right| = 1. \tag{204}$$

Equality (204) takes place for arbitrary $g^{(n)}$ and ε only for $|p| \leq 1/3$. For $p > 1/3$, the above sum depends on $g^{(n)}$ and ε and it is larger than one.

It is obvious if one calculates the tomogram using the element of the unitary group of the form

$$g_0^{(n)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{205}$$

At this point, the sum (204) reads

$$\sum_{m_1,m_2=-1/2}^{1/2} \left| \frac{1+\varepsilon}{4} - \varepsilon w_W(m_1, m_2, g^{(n)}) \right| = 3 \left| \frac{1+\varepsilon p}{4} \right| + \left| \frac{1-3p\varepsilon}{4} \right|. \tag{206}$$

One can see that this sum equals to one independently on the value of parameter $|\varepsilon| \leq 1$ only for values $|p| \leq 1/3$. For $p = 1$, the maximum value of the sum equals $2 = (1 + 3\varepsilon)/2$ ($\varepsilon = 1$). This value can characterize the degree of entanglement of Werner state.

We have introduced positive nonlinear map of density matrix which is purification map. The purification map can be combined with contractive maps discussed. The tomographic-probability distributions under discussion can be completely described by their characteristic functions. This means that the relation of tomogram property to entanglement can be formulated in terms of the properties of characteristic functions.

One can also check the criterion using example of two-qutrite pure entangled state with wave function

$$| \psi\rangle = \frac{1}{\sqrt{3}} \sum_{m=-1}^{1} | u_m\rangle \, | v_m\rangle. \tag{207}$$

The sum defining the criterion of separability for specific $U(9)$ transform $g_0^{(n)}$ which is diagonalizing the hermitian matrix $L_\varepsilon \mid \psi\rangle\langle\psi \mid$ reads

$$F(\varepsilon, g_0^{(n)}) = 8 \left| \frac{1+\varepsilon}{9} \right| + \left| \frac{1-8\varepsilon}{9} \right|. \tag{208}$$

For $1/2 > \varepsilon > 1/8$, this sum is larger than one, that means that the state is entangled. For $\varepsilon = 1/2$, the function has maximum and it is equal to $5/3$.

The entanglement of the considered state can be detected using partial transposition criterion too.

For the case of pure entangled state of two-qutrite system with the wave function

$$| \Phi\rangle = \frac{1}{\sqrt{2}} \Big(| u_1\rangle \, | v_1\rangle + | u_0\rangle \, | v_0\rangle \Big), \tag{209}$$

in which the states with spin projections $m = -1$ do not participate, the partial transpose criterion does also detect entanglement. Our criterion yields for specific $U(9)$ transform $g_0^{(n)}$, which diagonalizes the hermitian matrix $L_\varepsilon \, | \Phi\rangle\langle\Phi \, |$ the following expression for the function $F(\varepsilon, g_0^{(n)})$, which reads

$$F(\varepsilon, g_0^{(n)}) = 5\, \frac{|1+\varepsilon|}{6} + \frac{|1-5\varepsilon|}{6}\,. \tag{210}$$

The function takes maximum value for $\varepsilon = 1/2$ that equals to $3/2$. This value is smaller than $5/3$ of the previous case. It corresponds to our intuition that the superposition of three product states of two qutrite system is more entangled than the superposition of only two such product states.

The criterion can be extended to multipartite spin system.

We have to apply for n-partite system the transform of the density matrix ρ of the form

$$L_{\vec{\varepsilon}} = L_{\varepsilon_1}^{(1)} \otimes L_{\varepsilon_2}^{(2)} \otimes \ldots \otimes L_{\varepsilon_n}^{(n)}, \tag{211}$$

where the transform $L_{\varepsilon_k}^{(k)}$ acts as depolarizing map on the kth subsystem. If the state is separable

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(n)}, \quad \sum_k p_k = 1, \quad p_k \geq 0, \tag{212}$$

each of the terms $\rho_k^{(j)}$ $(j = 1, 2, \ldots, n)$ in the tensor product is replaced by the term

$$\rho_k^{(j)} \to -\varepsilon_j \rho_k^{(j)} + \frac{1+\varepsilon_j}{N_j}\, 1_j. \tag{213}$$

This means that the transformed density matrix reads

$$\rho \to L_{\vec{\varepsilon}} \rho = \sum_k p_k \left[\prod_{j=1}^{n} \otimes \left(-\varepsilon \rho_k^{(j)} + \frac{1+\varepsilon_j}{N_j}\, 1_j \right) \right]. \tag{214}$$

The unitary spin tomogram of the transformed density matrix takes the form $(\vec{\varepsilon} = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$

$$w_{\vec{\varepsilon}}(m_1, m_2, \ldots, m_n, g^{(N)}) = \sum_k p_k w_{pr}^{(k)}(m_1, m_2, \ldots, m_n, g^{(N)}, \vec{\varepsilon}), \tag{215}$$

where $N = \prod_{s=1}^{n}(2j_s + 1)$ and element $g^{(N)}$ is the unitary matrix in N-dimensional space. The tomogram $w_{pr}^{(k)}(m_1, m_2, \ldots, m_n, g^{(N)}, \vec{\varepsilon})$ is the joint probability distribution of spin projections $m_s = -j_s, -j_s + 1, \ldots, j_s$, which depends on the unitary transform $g^{(N)}$ in the state with density matrix

$$\rho_k = \prod_{s=1}^{n} \otimes \left(-\varepsilon_s \rho_k^{(s)} + \frac{1 + \varepsilon_s}{N_s} 1_s \right). \tag{216}$$

For the elements

$$g_{pr}^{(N)} = \prod_{s=1}^{n} \otimes u_s(2j_s + 1),$$

where $u_s(2j_s+1)$ is unitary matrix, the tomogram (215) takes the form of sum of the products

$$w_{\vec{\varepsilon}}(m_1, m_2, \ldots, m_n, g_{pr}^{(N)}) = \sum_k p_k \prod_{s=1}^{n} w_k\Big(m_s, u_s(2j_s + 1), \varepsilon_s\Big), \tag{217}$$

with $w_k(m_s, u_s(2j_s + 1), \varepsilon_s)$ being the unitary spin tomograms of the sth spin subsystem with transformed density matrix $L_{\varepsilon_s}\rho_k^{(s)}$. If one uses as the matrix $u_s(2j_s + 1)$, the matrix of unitary irreducible representation of the $SU(2)$ group, the tomogram w_k depends only on the two parameters defining the point on the sphere S^2.

For a separable state of the multipartite system, one has

$$\sum_{m_1,\ldots,m_n} \left| w_{\vec{\varepsilon}}(m_1, m_2, \ldots, m_n, g^{(N)}) \right| = 1 \tag{218}$$

for all elements $g^{(N)}$ and all parameters $\vec{\varepsilon}$.

For entangled state, there can be some values of parameters $\vec{\varepsilon}$ and group elements $g^{(N)}$ for which the sum is larger than one.

12. Conclusions

We summarize the results of the paper.

The notion of entangled states (first discussed by Schrödinger [4, 51]) has attracted a lot of efforts to find a criterion and quantitative characteristics of entanglement. A criterion based on partial transpose transform of subsystem density matrix (complex conjugation of the subsystem density matrix or its time reverse) provides the necessary and sufficient condition of separability of the system of two qubits and qubit-qutritt system [52]. The phase-space representation of the quantum states and time reverse transform (change of the signs of the

subsystem momenta) of the Wigner function in the case of Gaussian state was applied to study the separability and entanglement of photon states in [13]. Recently it was pointed out that the tomographic approach of reconstructing the Wigner function of quantum state [43–45] can be developed to consider the positive probability distribution (tomogram) as an alternative to density matrix (or wave function) because the complete set of tomograms contains the complete information on the quantum state [42]. This representation (called probability representation) was constructed also for spin states including a bipartite system of two spins. Up to now the problem of entanglement was not discussed in the tomographic representation. Some remarks on tomograms and entanglement of photon states in the process of Raman scattering were done in [53]. The tomographic approach has the advantage of dealing with positive probabilities and one deals with standard probability distributions which are positive and normalized.

We studied the properties of separable and entangled state of multipartite system using the tomographic probability distributions. The positive and completely positive maps of density matrices [39, 54] induce specific properties of the tomograms. The properties of the positive maps were studied in [55]. We formulated necessary and sufficient conditions of separability and entanglement of multipartite systems in terms of properties of the quantum tomogram. Since the tomograms were shown [36] to be related to the star-product quantization procedure [56], we discuss entanglement and separability properties in terms of generic operator symbols. The tomographic symbols of generic spin operators were studied in [36]. Then we focused on properties of entanglement and separability of a bipartite system using spin tomograms ($SU(2)$-tomograms) and tomograms of the $U(N)$-group.

The idea of the approach suggested can be summarized as following.

The positive but not completely positive linear maps of a subsystem density matrix do preserve the positivity of separable density matrix of the composite system. These maps contain also maps which do not preserve the positivity of the initial density matrix of an entangled state for the composite system. It means that the set of all linear positive maps of the subsystem density matrix (this set is semigroup) creates from the initial entangled positive density matrix of composite system a set of hermitian matrices including the matrices with negative eigenvalues. To detect the entanglement we use the tomographic symbols of the obtained hermitian matrices.The tomographic symbols of state density matrices (state tomograms)are standard probabilities.In view of this the tomographic symbols of the obtained hermitian matrices corresponding to initial separable state preserve all the properties of the probability representation including positivity and normalization.

But in case of entangled state the tomographic symbols of the obtained hermitian matrices can take negative values.The different behaviour of tomograms of separable and entangled states of composite systems under action of the semigroup of positive maps of the subsystem density matrix provides the tomographic criterion of the separability.

To conclude, we point out the main result of the work.

We found the criterion of separability which is given by equation (173). The criterion is valid for multiparticle spin system. The criterion can be called "tomographic criterion" of separability. The tomographic criterion can be considered also for symplectic tomograms of multimode photon states. The condition of separability is sufficient because there always exists a unitary group element by means of which any hermitian matrix can be diagonalized.It means that tomographic symbol of nonpositive hermitian matrix has nonpositive values for some unitary group parameters. The suggested criterion is connected with properties of the constructed function (173) which for given density matrix depends on unitary group parameters g and the parameters of positive map semigroup L. For separable density matrix the dependence on unitary group parameters and the semigroup parameters disappears and the function becomes constant equal to unity. For entangled states the function differs from unity and depends on both group and semigroup parameters. The suggested criterion can be considered as some complementary test of separability together with other criteria available in the literature (see, for example, [38, 52]). We point out that suggested criterion differs from available usual ones by the kind of the necessary numerical calculations. To apply this criterion one needs to calculate the sum of moduli of diagonal matrix elements of product of three matrices. One of the matrices is hermitian and two others are unitary ones. This procedure does not need the calculation of the eigenvalues of a matrix. The structure of positive (including not completely positive) map semigroup with elements L needs extra investigation (see, for example, [57]). We found also a test of entanglement based on the property of unitary spin tomogram.

The discussed purification map can be applied to find new quantum evolution equations in addition to known ones [58–61]. The application of different forms of positive maps [55, 62] and supermatrix representation of the maps [63, 64] are useful for better understanding of the computations. Entanglement phenomena can be considered using symbols of density matrix of different kinds, e.g., particular quasidistributions [65] as well as tomographic symbols [36]. The difference of symbols of entangled and separable density operators for different schemes of the star-product quantization needs further investigations as well as test of entanglement of some generalizations of Werner state [66, 40]

in multipartite case. A relation of tomographic approach to different positive maps [67] should be investigated. The tomographic symbols are analytic in group parameters. This can be used to find extrema of tomograms which give information on degree of entanglement.

Acknowledgments

V. I. M. and E. C. G. S. thank Dipartimento di Scienze Fisiche, Universitá "Federico II" di Napoli and Istitito Nazionale di Fisica Nucleare, Sezione di Napoli for kind hospitality. V. I. M. is grateful to the Russian Foundation for Basic Research for partial support under Project No. 01-02-17745.

References

[1] P. A. M. Dirac 1958 "The Principles of Quantum Mechanics" (Oxford: Pergamon)
[2] L. D. Landau 1927 *Z. Phys* **45** 430
[3] J. von Neumann 1932 "Mathematische Grudlagen der Quantenmechanik" (Berlin: Springer); Nov. 1927 *Göttingenische Nachrichten* **11** S245
[4] E. Schrödinger 1935 *Naturwissenschaften* **23** 807; 823; 844
[5] E. Schrödinger 1926 *Ann. d. Phys.* Lpz **79** 489
[6] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria 1999 *J. Russ. Laser Res.* **20** 421; 2002 *J. Phys. A: Math. Gen.* **35** 7173
[7] M. Horodecki, P. Horodecki and R. Horodecki 1996 *Phys. Lett.* A **223** 1
[8] S. Hill and W. K. Wootters 1997 *Phys. Rev. Lett.* **78** 5022
W. K. Wootters 1998 *Phys. Rev. Lett.* **80** 2245
[9] K. Zyczkowski, P. Horodecki, A. Sanpera and M. Lewenstein 1998 *Phys. Rev.* A **58** 883
[10] S. Popescu and D. Rohrlich 1997 *Phys. Rev.* A **56** R3319
[11] S. Abe and A. K. Rajagopal 2002 *Physica* A **289** 157
[12] C. H. Bennett, D. P. Di Vincenzo, J. A. Smolin and W. L. Wootters 1996 *Phys. Rev.* A **54** 3824
[13] R. Simon 2002 *Phys. Rev. Lett.* **84** 2726
[14] S. M. Barnett and S. J. D. Phoenix 1989 *Phys. Rev.* A **40** 2404; 1991 ibid **44** 535
[15] A. Mann, B. C. Sanders and W. J. Munro 1995 *Phys. Rev.* A **51** 989
[16] C. H. Bennet, H. J. Bernstein, S. Popescu and B. Schumacher 1996 *Phys. Rev.* A **53** 2046
[17] V. Vedral and M. B. Plenio 1998 *Phys. Rev.* A **57** 1619
[18] B.-G. Englert, M. Löffler, O. Benson, B. Varcoe, M. Weidinger and H. Walter 1998 *Fortschr. Phys.* **46** 897
[19] M. Horodecki, P. Horodecki and R. Horodecki 2000 *Phys. Rev. Lett.* **84** 2014
[20] S. Parker, S. Bose and M. B. Plenio 2000 *Phys. Rev.* A **61** 032305
[21] C. H. Bennet, S. Popescu, D. Rohrlich, J. A. Smolin and A. V. Thapliyal 2001 *Phys. Rev.* A **63** 012307
[22] K. Piątek and W. Leoński 2001 *J. Phys. A: Math. Gen.* **34** 4951

[23] K. Audenaert, J. Esert, E. Jané, M. B. Plenio, S. Virmani and R. B. De Moor 2001 *Phys. Rev. Lett.* **87** 217902
[24] K. Furuya, M. C. Nemes and G. Q. Pellegrino 1998 *Phys. Rev. Lett.* **80** 5524
[25] K. Życzkowski, P. Horofecki, A. Savpera and M. Lewenstein 1998 *Phys. Rev.* A **58** 883
[26] W. J. Munro, D. F. V. James, A. G. White and P. G. Kwait 2001 *Phys. Rev.* A **64** 030302
[27] G. Vidal and R. F. Werner 2002 *Phys. Rev.* A **65** 032314
[28] F. Coffman, J. Kundu and W. K. Wootters 2000 *Phys. Rev.* A **61** 052306
[29] P. Badziag, P. Deuar, M. Horodecki, P. Horodecki and R. Horodecki 2002 *J. Mod. Opt.* **49** 1289
[30] M. A. Andreata, A. V. Dodonov and V. V. Dodonov 2002 *J. Russ. Laser Res.* **23** 531
[31] V. V. Dodonov and V. I. Man'ko 1997 *Phys. Lett.* A **229** 335
[32] Olga Man'ko and V. I. Man'ko 1997 *JETP* **85** 430
[33] A. B. Klimov, O. V. Man'ko, V. I. Man'ko, Yu. F. Smirnov and V. N. Tolstoy 2002 *J. Phys. A: Math. Gen.* **35** 6101
[34] V. A. Andreev and V. I. Man'ko 1998 *JETP* **87** 239
[35] V. I. Man'ko and S. S. Safonov 1998 *Yad. Fiz.* **61** 658
[36] O. V. Man'ko, V. I. Man'ko and G. Marmo 2000 *Phys Scr.* **62** 446; 2002 *J. Phys. A: Math. Gen.* **35** 699
[37] V. V. Dodonov, A. S. M. De Castro and S. S. Misrahi 2002 *Phys. Lett.* A **296** 73
A. S. M. De Castro and V. V. Dodonov 2003 *J. Russ. Laser Res.* **23** 93; 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** S593
[38] Special Issue on Entanglement 2002 *J. Math. Phys.* **43** No. 9
[39] E. C. G. Sudarshan, P. M. Mathews and J. Rau 1961 *Phys. Rev.* **121** 920
[40] E. C. G. Sudarshan and A. Shaji 2003 "Structure and parametrization of stochastic maps of density matrix" ArXiv quant-ph/0205051 v2; 2003 *J. Phys. A: Math. Gen.* **36** (in press)
[41] A. Peres 1996 *Phys. Rev. Lett.* **77** 1413
[42] S. Mancini, V. I. Man'ko and P. Tombesi 1996 *Phys. Lett.* A **213** 1; 1997 *Found. Phys.* **27** 801
[43] J. Bertrand and P. Bertrand 1987 *Found. Phys.* **17** 397
[44] K. Vogel and H. Risken 1989 *Phys. Rev.* A **40** 2847
[45] S. Mancini, V. I. Man'ko and P. Tombesi 1995 *Quantum Semiclass. Opt.* **7** 615
G. M. D'Ariano, S. Mancini, V. I. Man'ko and P. Tombesi 1996 *Quantum Semiclass. Opt.* **8** 1017
[46] M. A. Man'ko, V. I. Man'ko and R. V. Mendes 2001 *J. Phys. A: Math. Gen.* **24** 8321
[47] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria 2003 "Entanglement in probability representation of quantum states and tomographic criterion of separability" *J. Opt. B: Quantum Semiclass. Opt.* (in press); 2003 *J. Russ. Laser Res.* **24** 507
[48] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria 2000 "Inner composition law of pure-spin states" in "Spin-Statistics Connection and Commutation Relations" R. C. Hilborn and G. M. Tino (eds.) *AIP Conference Proceedings* **545** 92
[49] A. S. Holevo 1999 *Russ. Math. Surveys* **53** 1295
[50] P. W. Shor 2002 *J. Math. Phys.* **43** 4334
[51] E. Schrödinger 1935 *Proc. Cambridge Philos. Soc.* **31** 555

[52] P. B. Slater 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** S691
[53] S. V. Kuznetsov, O. V. Man'ko and N. V. Tcherniega 2003 *J. Opt. B: Quantum Semiclass. Opt.* **6** S503
[54] W. F. Stinespring 1955 *Proc. Amer. Math. Soc.* **6** 211
[55] L. C. Woronowicz 1976 *Rep. Math. Phys.* **10** 165
[56] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerovicz and D. Sternheimer 1975 *Lett. Math. Phys.* **1** 521
[57] C. King 2002 "The capacity of the quantum depolarizing channel" ArXiv quant-ph/0204172 v2
[58] J. E. Moyal 1949 *Proc. Cambridge Philos. Soc.* **45** 99
[59] A. Kossakovski 1972 *Rep. Math. Phys.* **3** 247
[60] G. Lindblad 1976 *Comm. Math. Phys.* **48** 119
[61] V. Gorini, A. Kossakovski and E. C. G. Sudarshan 1978 *Rep. Math. Phys.* **18** 149
[62] K. Kraus 1973 *Ann. Phys.* NY **64** 311
[63] M. D. Choi 1975 *Lin. Alg. Appl.* **10** 285; 1970 *Canadian J. Math.* **24** 520; 1976 *Illinois J. Math.* **48** 119
[64] T. F. Havel 2003 *J. Math. Phys.* **44** 534
[65] E. C. G. Sudarshan 1963 *Phys. Rev. Lett.* **10** 277
C. L. Mehta and E. C. G. Sudarshan 1965 *Phys. Rev.* B **138** 274
[66] R. Werner 1989 *Phys. Rev.* A **40** 4277
[67] A. Kossakowski 2003 "A class of linear positive maps in matrix algebras" ArXiv quant-ph/0307132 v1